Parameterized Supply Function Equilibrium in Power Networks

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Abstract—**We consider the setting in which generators compete in scalar-parameterized supply functions to serve an inelastic demand spread throughout a transmission constrained power network. The market clears according to a locational marginal pricing mechanism, in which the independent system operator (ISO) determines the generators' production quantities so as to minimize the revealed cost of meeting demand, subject to transmission and generator capacity constraints. Under the assumption that both the ISO and generators choose their strategies simultaneously, we establish the existence of Nash equilibria for the underlying game, and derive a tight bound on its price of anarchy. Under the more restrictive setting of a two-node power network, we present a detailed comparison of market outcomes predicted by the simultaneous-move formulation of the game against those predicted by the more plausible sequential-move formulation, where the ISO observes the generators' strategy profile prior to determining their production quantities.**

I. Introduction

We consider an electricity market design in which power producers compete in supply functions to meet an inelastic demand distributed throughout a transmission constrained power network. Such markets are susceptible to manipulation given the large leeway afforded producers in reporting their supply functions [1]. The potential for market manipulation is amplified by the largely inelastic nature of electricity demand and the presence of network transmission constraints [2], [3]. For example, the market manipulation underlying the 2000-01 California electricity crisis resulted in over 40 billion US dollars in excessive energy cost, and the bankruptcy of PG&E [4]. In this paper, we consider the setting in which producers are required to bid supply functions belonging to a scalar-parameterized family [5], [6]. Our primary goal is to characterize and bound the welfare loss that might emerge due to the strategic interactions between producers and the independent system operator (ISO) in this setting. In doing so, we aim to identify the way in which transmission capacity constraints might influence the extent to which producers can exercise market power.

Related Work and Contribution: The study of supply function equilibria dates back to the seminal work of Klemperer and Meyer [1], which revealed that, in the absence of demand uncertainty, nearly any market outcome can be supported by a supply function equilibrium. There has subsequently emerged a large body of literature employing similar models of supply function equilibria as a means to analyze competition in electricity markets [7]–[12]. More recently, there has been a growing interest in quantifying the quality of a supply function equilibrium under simplifying assumptions on the functional form of the supply functions. A scalar-parameterized supply function bidding mechanism was proposed and analyzed by Johari and Tsitsiklis [5], in which N producers compete to meet an inelastic demand. Given the assumption that each producer is able to meet the demand individually, they show that the price of anarchy is upper bounded by $1 + 1/(N - 2)$. Xu et al. [6] extend these results to the setting in which producers encode their production capacities in the supply functions they bid. The efficiency loss incurred at linear supply function equilibria has also been studied in [13], [14].

While Klemperer's supply function model offers a compelling description of competition in electricity markets, the characterization and analysis of supply function equilibria becomes challenging in the presence of network transmission constraints [15], [16]. For example, it is well known that if we restrict ourselves to linear or piecewise-constant supply functions, supply function equilibria may fail to exist in simple two or three-node networks [17], [18]. In an effort to address such difficulties, there has emerged another stream of literature that resorts to the so-called networked Cournot models to characterize the strategic interaction between producers in constrained transmission networks. We refer the reader to [19]–[25] for recent advances.

In this paper, we build on the recent literature [5], [6] to develop a rigorous equilibrium analysis of a locational marginal pricing mechanism in which generators are required to report scalar-parameterized supply functions. We note that this is in contrast to the uniform pricing mechanism considered in [14]. Adopting a solution concept in which the ISO and generators choose their strategies simultaneously, we derive an upper bound on the worst-case efficiency loss incurred at a Nash equilibrium, and identify conditions under which this bound is guaranteed to be tight.

Organization: In Section II, we introduce the scalar-parameterized supply function bidding mechanism, and formulate the networked supply function game. Section III establishes the existence of Nash equilibria, and provides an upper bound on their worst-case efficiency loss. In Section IV, we compare the simultaneous-move and sequential-move formulations of the supply function game in terms of the market equilibria they predict in a two-node power network. Section V concludes the paper with directions for future research. All proofs are omitted due to space constraints.

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Notation: Let $\mathbb{R}$ denote the set of real numbers, and $\mathbb{R}_+$ the set of non-negative real numbers. Denote the transpose of a vector $x \in \mathbb{R}^n$ by $x^\top$. Let $x_{-i} = (x_1, .., x_{i-1}, x_{i+1}, .., x_n) \in \mathbb{R}^{n-1}$ be the vector including all but the i^{th} element of x. Denote by $\mathbf{1}$ the vector of all ones. Denote by $x \mapsto [x]_a^b$ the mapping, which projects $x \in \mathbb{R}$ onto the closed interval $[a, b]$.

II. Model and Formulation

A. Supply and Demand Models

We consider the setting in which producers compete to supply energy to an inelastic demand spread throughout a transmission constrained power network. The network is assumed to have a connected topology consisting of n transmission buses (or nodes) connected by m transmission lines (or edges). Let $\mathcal{V} := \{1, \ldots, n\}$ denote the set of all nodes. In addition, we assume that there are N_i producers located at each node $i \in \mathcal{V}$, and denote by $N := \sum_{i=1}^n N_i$ the total number of producers. We specify the nodal position of each producer according to an incidence matrix $A \in \{0, 1\}^{n \times N}$, defined as

$$A_{ij} := \begin{cases} 1, & \text{if producer } j \text{ is located at node } i, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let $\mathcal{N}_i := \{j \,|\, A_{ij} = 1\}$ be the set of producers at node i.

Demand is assumed to be inelastic. Accordingly, we let $d \in \mathbb{R}_+^n$ represent the demand profile across the network, where d_i denotes the demand for energy at node i. We let x_j be the production quantity of producer j, and denote by $C_j(x_j)$ his cost of producing x_j units of energy. We denote by $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$ the production profile, and by $C := (C_1, \ldots, C_N)$ the cost function profile. We make the following standard assumption regarding the producers' cost functions.

Assumption 1. For each producer $j \in \{1, \ldots, N\}$, his production cost $C_j(x_j)$ is a convex function that satisfies $C_j(x_j) = 0$ for $x_j \leq 0$, and $C_j(x_j) > 0$ for $x_j > 0$.

For each producer $j \in \{1, \ldots, N\}$, we let $X_j \geq 0$ denote his maximum production capacity.

B. The Economic Dispatch Problem

Ultimately, the objective of the independent system operator (ISO) is to choose a production profile that minimizes the true cost of serving the demand, while respecting the capacity constraints on transmission and generation facilities. Doing so amounts to solving the so called *economic dispatch* (ED) problem, which is formally defined as

$$\begin{aligned} \underset{x \in \mathbb{R}^N}{\text{minimize}} \quad & \sum_{j=1}^N C_j(x_j) \\ \text{subject to} \quad & Ax - d \in \mathcal{P}, \\ & 0 \leq x_j \leq X_j, \; j = 1, \ldots, N, \end{aligned} \tag{2}$$

where $\mathcal{P}$ represents the feasible set of nodal power injections over the network. Adopting the assumptions that underlie the so called *DC power flow model* [26], one can represent the set $\mathcal{P}$ as a polytope

$$\mathcal{P} = \left\{ y \in \mathbb{R}^n \;\middle|\; \mathbf{1}^\top y = 0, \; Hy \leq c \right\},$$

where $H \in \mathbb{R}^{2m \times n}$ denotes the shift-factor matrix, and $c \in \mathbb{R}^{2m}$ the corresponding vector of transmission line capacities. We will refer to the constraint $\mathbf{1}^\top y = 0$ as the *power balance constraint*, and the constraint $Hy \leq c$ as the *transmission capacity constraint*. Any production profile $x^* = (x_1^*, \ldots, x_N^*) \in \mathbb{R}^N$ that solves (2) is called *efficient*.

C. Scalar-parameterized Supply Function Bidding

In practice, the ISO does not have access to the producers' true cost information. Instead, the producers are asked to report their private information to the ISO in the form of supply functions, which specify the maximum quantity a producer is willing to supply at a particular price. A basic challenge in the design of such markets resides in the choice of the class of functions from which a producer is allowed to select its supply function. In principle, the class of functions should be rich enough to allow for the accurate reporting of a producer's true cost information, but not so rich as to allow for the excessive exercise of market power. For instance, it has been shown that such markets can exhibit unbounded efficiency loss, if producers are allowed to bid arbitrary supply functions, or other natural parametric forms (e.g., linear or piecewise-constant) [1], [13], [17]. In what follows, we investigate the setting in which producers are allowed to bid *scalar-parameterized supply functions*, and analyze the existence and efficiency of market equilibria that result under price-anticipating producer behavior.

In particular, we adopt the approach of Xu et al. [6], and consider a capacitated version of Johari and Tsitsiklis's scalar-parameterized supply function [5]. Specifically, each producer j reports a scalar parameter $\theta_j \in \mathbb{R}_+$ that defines a supply function of the form

$$S_j(p; \theta_j) = X_j - \frac{\theta_j}{p}, \tag{3}$$

where $S_j(p; \theta_j)$ denotes the maximum quantity that producer j is willing to supply at a price $p > 0$. Here, X_j is the *true* production capacity of producer j. We do not allow producers to bid their capacities strategically, as this would typically incur a large efficiency loss at a supply function equilibrium [9]. We denote the strategy profile of all producers by $\theta := (\theta_1, \ldots, \theta_N) \in \mathbb{R}_+^N$.

Remark 1 (Negative Supply)**.** A practical drawback of the class of supply functions we consider is that they allow for the possibility of market outcomes in which a producer has negative output. We will, however, show that such outcomes are not possible at equilibrium. Namely, the output of a producer is guaranteed to be non-negative at the Nash equilibrium. Moreover, one can show that the results of this paper are preserved under a modified class of supply functions given by $S_j(p; \theta_j) = \max\left\{X_j - \frac{\theta_j}{p}, \; -\epsilon\right\}$, where $\epsilon > 0$ is an arbitrary positive constant. We avoid this alternative treatment to facilitate ease of exposition in the present paper.

Given the producers' strategy profile θ, the ISO's objective is to obtain an allocation that minimizes the reported aggregate production cost, subject to the network transmission and production capacity constraints. The *reported cost function* of producer j is defined as the integral of its inverse supply function, which is given by

$$\widehat{C}_j(x;\theta_j) := \int_0^x \frac{\theta_j}{X_j - z} dz \ = \ \theta_j \log\left(\frac{X_j}{X_j - x}\right). \quad (4)$$

With these reported costs in hand, the ISO solves the following economic dispatch (ED) problem:

$$\begin{aligned} \underset{x\in\mathbb{R}^N}{\text{minimize}} \quad & \sum_{j=1}^{N} \widehat{C}_j(x_j;\theta_j) \\ \text{subject to} \quad & Ax - d \in \mathcal{P}, \\ & x_j \le X_j, \ j = 1,\dots,N. \end{aligned} \quad (5)$$

Note that we have dropped the non-negativity constraints on supply, as the class of supply functions (3) we consider allow for negative supply.

D. A Primal Decomposition of Economic Dispatch

In what follows, we develop a primal decomposition of the ED problem (5), which reveals an explicit relationship between an individual producer's production quantity and the aggregate production quantity at his node. We do so through introduction of an auxiliary variable $q := Ax \in \mathbb{R}^n$, which we refer to as the *nodal supply profile*. Here, $q_i = \sum_{j\in\mathcal{N}_i} x_j$ represents the aggregate production quantity at node i, and serves as the coupling variable between the network-wide ED problem and the nodal ED problem in the local variables $\{x_j | j \in \mathcal{N}_i\}$ at node i. More formally, the ED problem (5) admits an equivalent reformulation:

$$\begin{aligned} \underset{q\in\mathbb{R}^n}{\text{minimize}} \quad & \sum_{i=1}^{n} G_i(q_i;\theta) \\ \text{subject to} \quad & q - d \in \mathcal{P}, \\ & q_i = 0, \qquad \text{if } N_i = 0, \\ & q_i \le \sum_{j\in\mathcal{N}_i} X_j, \quad \text{if } N_i > 0, \quad i = 1,\dots,n, \end{aligned} \quad (6)$$

where $G_i(q_i;\theta)$ denotes the optimal value of the ED problem local to node i, and is defined as

$$G_i(q_i;\theta) := \min\left\{ \sum_{j\in\mathcal{N}_i} \widehat{C}_j(x_j;\theta_j) \,\middle|\, \sum_{j\in\mathcal{N}_i} x_j = q_i, \ x_j \le X_j, \ \forall j \in \mathcal{N}_i \right\}. \quad (7)$$

Each of the local ED problems in (7) admits closed-form solutions for its local variables $\{x_j | j \in \mathcal{N}_i\}$ in terms of the coupling variable q_i. If $\sum_{j\in\mathcal{N}_i} \theta_j > 0$, then the unique optimal solution to (7) is given by

$$x_j(q_i,\theta) = X_j - \frac{\theta_j}{\sum_{k\in\mathcal{N}_i}\theta_k} \cdot \left(\left(\sum_{k\in\mathcal{N}_i} X_k \right) - q_i \right). \quad (8)$$

for all $j \in \mathcal{N}_i$. On the other hand, if $\sum_{j\in\mathcal{N}_i} \theta_j = 0$, an optimal solution to (7) is given by

$$x_j(q_i,\theta) = \frac{X_j}{\sum_{k\in\mathcal{N}_i} X_k} \cdot q_i, \quad (9)$$

for all $j \in \mathcal{N}_i$. With equations (8) - (9) in hand, a closed-form expression for the production cost at node i is given by $G_i(q_i;\theta) = \sum_{j\in\mathcal{N}_i} \widehat{C}_j(x_j(q_i,\theta);\theta_j)$.

Remark 2 (Local Strategies)**.** We note that $x_j(q_i,\theta)$ depends on the global strategy profile θ only through the *local strategy profile* $\{\theta_k | k \in \mathcal{N}_i\}$. This reveals an important insight. Namely, given a fixed nodal supply profile q, the supply function game between producers across the network decouples into n disjoint games. Such insight will play a central role in our game theoretic analysis in Section III.

E. Networked Supply Function Game

We proceed with the development of a formal model of competition in supply functions over the power network. We denote the set of players as $\mathcal{N} := \{0,1,\dots,N\}$, where 0 denotes the ISO, and $j \in \{1,\dots,N\}$ denotes the jth producer. In practice, the producers and ISO engage in a *sequential-move* game in which the producers simultaneously report their bids, in anticipation of the ISO's determination of production quantities and nodal prices according to the solution of the ED problem (5). However, given the generality of the setting considered in this paper, a general equilibrium analysis of a sequential-move formulation is seemingly out of reach. We thus make a simplifying assumption, and adopt a model of competition, which assumes that the producers and ISO *move simultaneously*. We remark that such assumption of simultaneous movement is common in the literature on networked Cournot games (cf. [19]–[23]). We discuss the potential ramifications of such assumption in Section IV, through analysis of a simple two-node power network. We proceed with a formal description of the market participants, their strategy sets, and payoff functions.

Independent System Operator: The ISO chooses the production quantities of the individual producers to minimize the reported aggregate cost, while respecting transmission and production capacity constraints. Given the primal decomposition of the ED problem developed in Section II-D, such choice can be reduced to the determination of the nodal supply profile $q \in \mathbb{R}^n$ – which we take to be the strategy of the ISO. Consequently, we define the *payoff of the ISO* as

$$\pi_0(q,\theta) := -\sum_{i=1}^{n} G_i(q_i;\theta),$$

where his feasible strategy set is defined as

$$\mathcal{X}_0 := \left\{ q \in \mathbb{R}^n \,\middle|\, q - d \in \mathcal{P}, \ q_i \le \sum_{j\in\mathcal{N}_i} X_j, \ \text{if } N_i > 0, \ q_i = 0, \ \text{if } N_i = 0 \right\}.$$

Producers: For each node $i \in \mathcal{V}$, each producer $j \in \mathcal{N}_i$ decides on a bid parameter $\theta_j \geq 0$, which specifies his supply function. The production quantity of producer j is given by $x_j(q,\theta) := x_j(q_i,\theta)$, where the right-hand side is specified according to Equations (8)–(9).

Prices are allowed to vary across nodes. In particular, the price $p_i(q,\theta)$ at node i is chosen to *clear the market at that node.* If $\sum_{j\in\mathcal{N}_i}\theta_j > 0$, such price is the unique solution to the equation $\sum_{j\in\mathcal{N}_i} S_j(p;\theta_j) = q_i$, which is given by:

$$p_i\left(q,\theta\right) = \frac{\sum_{j\in\mathcal{N}_i}\theta_j}{\left(\sum_{j\in\mathcal{N}_i}X_j\right) - q_i}, \qquad \text{if } \sum_{j\in\mathcal{N}_i}\theta_j > 0. \quad (10)$$

On the other hand, if $\sum_{j\in\mathcal{N}_i}\theta_j = 0$, then we have that $S_j(p;\theta_j) = X_j$ for all $j \in \mathcal{N}_i$ whatever the price p. In this case, we set the price equal to zero:

$$p_i\left(q,\theta\right) = 0, \qquad \text{if } \sum_{j\in\mathcal{N}_i}\theta_j = 0. \quad (11)$$

With the preceding specification of production quantity and price in hand, the *payoff of producer* $j \in \mathcal{N}_i$ is defined as

$$\pi_j\left(q,\theta\right) := p_i\left(q,\theta\right)x_j\left(q,\theta\right) - C_j\left(x_j\left(q,\theta\right)\right), \quad (12)$$

where his feasible strategy set is given by $\mathcal{X}_j := \mathbb{R}_+$.

Remark 3 (Locational Marginal Pricing)**.** When the nodal supply profile $q \in \mathbb{R}^n$ is chosen to solve the ED problem (6), the pricing mechanism specified in (10)–(11) corresponds to the so called *locational marginal pricing* mechanism used in many electricity markets that are in operation today. In the presence of transmission capacity constraints, such pricing mechanism ensures the existence of an efficient competitive equilibrium.

Solution Concept: Define $\mathcal{X} := \prod_{j=0}^{N}\mathcal{X}_j$ as the feasible strategy set for all players, and $\pi := (\pi_0,\pi_1,\ldots,\pi_N)$ as their collection of payoffs. The triple $(\mathcal{N},\mathcal{X},\pi)$ defines a normal-form game, which we shall refer to as the (networked) *simultaneous-move supply function game* for the remainder of this paper. We describe the stable outcome of the game $(\mathcal{N},\mathcal{X},\pi)$ according to Nash equilibrium.

Definition 1 (Nash Equilibrium)**.** The pair $(q,\theta) \in \mathcal{X}$ constitutes a *pure strategy Nash equilibrium* (NE) of the game $(\mathcal{N},\mathcal{X},\pi)$, if both of the following conditions are satisfied:

(i) $\pi_0\left(q,\theta\right) \geq \pi_0\left(\overline{q},\theta\right)$ for all $\overline{q} \in \mathcal{X}_0$,

(ii) $\pi_j\left(q,\theta_j,\theta_{-j}\right) \geq \pi_j\left(q,\overline{\theta}_j,\theta_{-j}\right)$ for all $\overline{\theta}_j \in \mathcal{X}_j$, $j = 1,\ldots,N$.

We let $\mathcal{X}_{\mathsf{NE}} \subseteq \mathcal{X}$ denote the set of all pure strategy Nash equilibria associated with the game $(\mathcal{N},\mathcal{X},\pi)$.

The production profile at a Nash equilibrium may differ from the efficient production profile. We use *price of anarchy* as a measure of the *efficiency loss* at a Nash equilibrium [27].

Definition 2 (Price of Anarchy)**.** The *price of anarchy* associated with the game $(\mathcal{N},\mathcal{X},\pi)$ is defined according to

$$\rho(\mathcal{N},\mathcal{X},\pi) := \sup\left\{ \frac{\sum_{j=1}^{N}C_j\left(x_j\left(q,\theta\right)\right)}{\sum_{j=1}^{N}C_j(x_j^*)} \,\middle|\, (q,\theta) \in \mathcal{X}_{\mathsf{NE}} \right\}.$$

III. Equilibrium Analysis

In this section, we characterize the Nash equilibrium of the supply function game. In a similar spirit to [5], [6], we characterize the production quantities of producers at a Nash equilibrium as the optimal solution to a parametric convex program. This facilitates both the proof of existence of a Nash equilibrium, and the derivation of upper bounds on the efficiency loss incurred at said equilibrium.

We begin by defining a quantity associated with the feasible injection polytope $\mathcal{P}$ and the demand profile d, which we refer to as the *maximum nodal supply*. In particular, the maximum nodal supply at node i is defined as

$$q_i^{\max} := \sup\left\{q_i \,\middle|\, q \in \mathbb{R}_+^n,\ q - d \in \mathcal{P}\right\}. \quad (13)$$

for each $i \in \mathcal{V}$. The vector $q^{\max} := (q_1^{\max},\ldots,q_n^{\max})$ will play an important role in guaranteeing the existence and in bounding the efficiency loss at a Nash equilibrium. Equipped with this concept, we present a basic assumption on the production capacities of all producers.

Assumption 2. For each node $i \in \mathcal{V}$ with $N_i > 0$, the following condition is satisfied

$$\sum_{k\in\mathcal{N}_i, k\neq j} X_k > q_i^{\max}, \quad \forall\, j \in \mathcal{N}_i.$$

Qualitatively, Assumption 2 requires that the removal of any single producer from a node does not preclude the remaining producers from meeting the maximum nodal supply at that node. Although Assumption 2 is not necessary for the existence of a Nash equilibrium, it will prove critical in guaranteeing the boundedness of the price of anarchy. In particular, it is straightforward to construct examples, which reveal that the efficiency loss at a Nash equilibrium can be arbitrarily large if Assumption 2 is violated.

Lemma 1 provides a characterization of Nash equilibrium through the solution of a parametric convex program.

Lemma 1 (Characterization of Nash Equilibrium)**.** Let Assumptions 1 - 2 hold. The pair $(q,\theta) \in \mathcal{X}$ constitutes a pure strategy Nash equilibrium of the game $(\mathcal{N},\mathcal{X},\pi)$ if and only if both of the following conditions are satisfied:

(i) The profile $x\left(q,\theta\right) := \left(x_1\left(q,\theta\right),\ldots,x_N\left(q,\theta\right)\right)$ given by (8)–(9) is the optimal solution to the following optimization problem parameterized by the nodal supply profile q:

$$\begin{aligned} \underset{x\in\mathbb{R}^N}{\text{minimize}} \quad & \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i}\tilde{C}_j(x_j;q_i) \\ \text{subject to} \quad & Ax - d \in \mathcal{P}, \\ & 0 \leq x_j \leq X_j, \quad j = 1,\ldots,N, \end{aligned} \quad (14)$$

where the modified cost function $\tilde{C}_j(x_j;q_i)$ is given by

$$\begin{aligned} \tilde{C}_j(x_j;q_i) = & C_j(x_j)\left(1 + \frac{x_j}{\sum_{k\in\mathcal{N}_i,k\neq j}X_k - q_i}\right) \\ & - \frac{1}{\sum_{k\in\mathcal{N}_i,k\neq j}X_k - q_i}\int_0^{x_j}C_j(z)\mathrm{d}z. \end{aligned} \quad (15)$$

(ii) The nodal supply profile q satisfies $q = Ax(q,\theta)$.

We illustrate the basic intuition behind the proof of Lemma 1 as follows. Given the assumption that the ISO moves simultaneously with all producers, one can "decouple" the supply function game $(\mathcal{N}, \mathcal{X}, \pi)$ over the network into n disjoint supply function bidding games similar in structure to those considered by [5], [6]. It is then a straightforward matter to establish a generalization of Theorem 4.1 in [6] that provides a characterization of Nash equilibrium in the form that we seek. The existence of a Nash equilibrium follows readily from the characterization in Lemma 1.

Proposition 1 (Existence of Nash Equilibrium)**.** Let Assumptions 1 - 2 hold. The game $(\mathcal{N}, \mathcal{X}, \pi)$ admits at least one pure strategy Nash equilibrium.

The following result guarantees that the efficiency loss at any Nash equilibrium is bounded from above.

Theorem 2 (PoA Bound)**.** Let Assumptions 1 - 2 hold. The price of anarchy (PoA) associated with the game $(\mathcal{N}, \mathcal{X}, \pi)$ is upper bounded by

$$\rho(\mathcal{N}, \mathcal{X}, \pi) \leq 1 + \max_{j \in \mathcal{N}_i, i \in \mathcal{V}} \left\{ \frac{\min\{X_j, q_i^{\max}\}}{\left(\sum_{k \in \mathcal{N}_i, k \neq j} X_k\right) - q_i^{\max}} \right\}.$$

Moreover, if there exists $i_0 \in \mathcal{V},\ j_0 \in \mathcal{N}_{i_0}$, such that

$$\max_{j \in \mathcal{N}_i, i \in \mathcal{V}} \left\{ \frac{\min\{X_j, q_i^{\max}\}}{\left(\sum_{k \in \mathcal{N}_i, k \neq j} X_k\right) - q_i^{\max}} \right\} = \frac{q_{i_0}^{\max}}{\left(\sum_{k \in \mathcal{N}_{i_0}, k \neq j_0} X_k\right) - q_{i_0}^{\max}}, \tag{16}$$

then the bound is tight. Namely, for any $\epsilon > 0$, there exists a cost function profile C^ϵ with a corresponding payoff profile of $\pi^\epsilon = (\pi_0^\epsilon, \cdots, \pi_N^\epsilon)$, such that

$$\rho(\mathcal{N}, \mathcal{X}, \pi^\epsilon) > 1 + \frac{q_{i_0}^{\max}}{\left(\sum_{k \in \mathcal{N}_{i_0}, k \neq j_0} X_k\right) - q_{i_0}^{\max}} - \epsilon.$$

The upper bound on the PoA in Theorem 2 depends on the nodal demand profile and the network transmission capacity constraints only through $q_i^{\max}$, the maximum nodal supply for each node i with $N_i > 0$. We note that the price of anarchy bound derived by Xiao et al. [14] under linear supply function bidding in power networks admits a similar structure. However, the tightness of their bound remains to be seen.

Theorem 2 reveals the possibility of a Braess-like paradox [28]. Specifically, an increase in a line's transmission capacity may result in an increase in the maximum nodal supply. This in turn may result in a greater loss of efficiency at a Nash equilibrium, as the PoA bound in Theorem 2 is strictly increasing in the maximum nodal supply. In Section IV, we verify the occurrence of such paradox in a two-node network with limited transmission capacity.

A. Bounding the Price Markup

Before concluding, we present a brief analysis related to the measure of a producer's market power at a Nash equilibrium. In particular, we use the *Lerner index* as our measure of a producer's market power [29], which is formally defined as follows. Given a nodal supply profile $q \in \mathcal{X}_0$ and producers' strategy profile $\theta \in \mathbb{R}_+^N$, for each node $i \in \mathcal{V}$, the Lerner index of each producer $j \in \mathcal{N}_i$ is defined as

$$L_j(q, \theta) := \frac{p_i(q, \theta) - \frac{\partial^+ C_j}{\partial x_j}(x_j(q, \theta))}{p_i(q, \theta)}.$$

where $\partial^+ C_j / \partial x_j (x_j(q, \theta))$ denotes the right derivative of the cost function C_j evaluated at $x_j = x_j(q, \theta)$. Essentially, the Lerner index measures the relative price markup above a producer's true marginal cost. The following corollary to Lemma 1 gives a bound on the Lerner index for each producer at a Nash equilibrium.

Corollary 1. Let Assumptions 1 - 2 hold. Given a Nash equilibrium (q, θ) of the game $(\mathcal{N}, \mathcal{X}, \pi)$, for each node $i \in \mathcal{V}$ and each producer $j \in \mathcal{N}_i$, the Lerner index of producer j is upper bounded by

$$L_j(q, \theta) \leq \frac{\min\{X_j, q_i^{\max}\}}{\sum_{k \in \mathcal{N}_i} X_k - \max\{X_j, q_i^{\max}\}}.$$

IV. Study of a Two-node Network

The equilibrium analysis presented in Section III relies on the assumption that both the ISO and power producers move simultaneously in determining their strategies. In this section, we present an alternative viewpoint, and analyze a sequential-move formulation of the supply function game, where the power producers choose their strategies simultaneously, in anticipation of the ISO's determination of the nodal supply profile according to the solution of the economic dispatch (ED) problem (6). In what follows, we restrict our analysis to the setting of a two-node power network, and present a detailed comparison of market outcomes predicted by the simultaneous-move formulation against those predicted by the more plausible sequential-move formulation of the supply function game.

A. System Description

Consider a two-node power network with a total demand of $D = 2$, and nodal demand profile given by $d = (D/2, D/2)$. We denote the capacity of the transmission line connecting the two nodes by $c \in \mathbb{R}_+$. The number of producers at nodes 1 and 2 are taken to be $N_1 = 3$ and $N_2 = 10$, respectively. We assume that each producer j has a production capacity of $X_j = 0.51D$.[1] We define the production cost functions as $C_j(x_j) = x_j$ for producers $j \in \mathcal{N}_1$, and as $C_j(x_j) = \beta x_j$ for producers $j \in \mathcal{N}_2$. We assume that $\beta > 1$. That is, the marginal cost of production is larger at node 2.

Due to the linearity of production costs and symmetry of producers at each node in the network, it is straightforward to show that the simultaneous-move game $(\mathcal{N}, \mathcal{X}, \pi)$ admits a unique Nash equilibrium (NE), which can be computed according to the necessary and sufficient conditions specified in Lemma 1.

[1] This choice of production capacities ensures that Assumption 2 is satisfied, thereby guaranteeing the existence of a NE under the simultaneous-move formulation.

B. The Sequential-move Formulation

In what follows, we develop and analyze the sequential-move formulation of the supply function game for the previously specified two-node network. In this setting, the ISO chooses the nodal supply profile q to solve the ED problem (6) after having observed the strategy profile θ reported by the producers; and all producers correctly anticipate the response of the ISO to their strategy profile.

We describe the resulting game between the producers in normal-form by explicitly encoding the ISO's response in the producers' payoff functions. Formally, we denote the set of all players (producers) by $\mathcal{N}^{\mathsf{seq}} := \{1, \ldots, N\}$, and define $\mathcal{X}^{\mathsf{seq}} := \prod_{j=1}^{N} \mathcal{X}_j$ as their set of feasible strategies. We denote the optimal solution to the ED problem (6) by $q(\theta) \in \mathbb{R}^2$. It is given by

$$q_1(\theta) = \left[\sum_{j \in \mathcal{N}_1} X_j - \frac{\sum_{j \in \mathcal{N}_1} \theta_j}{\sum_{k=1}^{N} \theta_k} \left(\sum_{k=1}^{N} X_k - D \right) \right]_{d_1 - c}^{d_1 + c},$$
$$q_2(\theta) = D - q_1(\theta).$$

One can therefore characterize the payoff of each producer j as an explicit function of the producers' strategy profile θ according to

$$Q_j(\theta) := \pi_j(q(\theta), \theta).$$

We define $Q := (Q_1, \ldots, Q_N)$ as the collection of payoff functions of producers. The triple $(\mathcal{N}^{\mathsf{seq}}, \mathcal{X}^{\mathsf{seq}}, Q)$ defines a normal-form game, which we refer to as the (networked) *sequential-move supply function game* for the remainder of this paper. In defining this normal-form game, we have explicitly encoded the ISO's response in each producer's payoff function, thereby capturing the sequential nature of the interaction between the ISO and producers. We describe the stable outcome of the game $(\mathcal{N}^{\mathsf{seq}}, \mathcal{X}^{\mathsf{seq}}, Q)$ according to Nash equilibrium.

Definition 3. The strategy profile $\theta \in \mathcal{X}^{\mathsf{seq}}$ constitutes a pure strategy Nash equilibrium of the game $(\mathcal{N}^{\mathsf{seq}}, \mathcal{X}^{\mathsf{seq}}, Q)$ if for each $j \in \mathcal{N}^{\mathsf{seq}}$ it holds that

$$Q_j(\theta_j, \theta_{-j}) \geq Q_j(\overline{\theta}_j, \theta_{-j}), \quad \text{for all } \overline{\theta}_j \in \mathcal{X}_j.$$

We let $\mathcal{X}_{\mathsf{NE}}^{\mathsf{seq}} \subseteq \mathcal{X}^{\mathsf{seq}}$ denote the set of all pure strategy Nash equilibria associated with the game $(\mathcal{N}^{\mathsf{seq}}, \mathcal{X}^{\mathsf{seq}}, Q)$.

It is worth mentioning that, for general networks, neither uniqueness nor existence of Nash equilibria is guaranteed under this (sequential-move) formulation of the supply function game. Implicit in such difficulty is the fact that, in general, a producer's payoff function may fail to be quasi-concave in his strategy. Nevertheless, one can characterize a pair of necessary conditions for Nash equilibrium (NE) given the two-node power network under investigation in this section as follows. Fix a strategy profile $\theta \in \mathcal{X}_{\mathsf{NE}}^{\mathsf{seq}}$. If the transmission line is congested at the corresponding nodal supply profile $q(\theta)$, then $(q(\theta), \theta)$ is the unique NE of the simultaneous-move game $(\mathcal{N}, \mathcal{X}, \pi)$. If, on the other hand, the transmission line is uncongested at the corresponding nodal supply profile $q(\theta)$, then one can show that θ equals the unique NE of a *modified* supply function game, in which all N producers compete for an inelastic demand of D in the absence of network transmission constraints.[2] Finally, these necessary conditions yield a finite set of two candidate strategy profiles, each of which can be individually evaluated to verify as to whether it is indeed a NE, or not. If both strategy profiles fail to be a NE, then the set $\mathcal{X}_{\mathsf{NE}}^{\mathsf{seq}}$ is necessarily empty.

We define the price of anarchy associated with the sequential-move game $(\mathcal{N}^{\mathsf{seq}}, \mathcal{X}^{\mathsf{seq}}, Q)$ according to

$$\rho(\mathcal{N}^{\mathsf{seq}}, \mathcal{X}^{\mathsf{seq}}, Q) := \sup \left\{ \frac{\sum_{j=1}^{N} C_j\left(x_j\left(q(\theta), \theta\right)\right)}{\sum_{j=1}^{N} C_j(x_j^*)} \,\middle|\, \theta \in \mathcal{X}_{\mathsf{NE}}^{\mathsf{seq}} \right\}.$$

The price of anarchy is left undefined if $\mathcal{X}_{\mathsf{NE}}^{\mathsf{seq}}$ is empty.

C. Discussion

In Figures 1a-1b, we plot the price of anarchy predicted by the simultaneous-move and sequential-move formulations as a function of the transmission line capacity c. When the transmission capacity c is small, the price of anarchy under the simultaneous-move and sequential-move formulations of the game are identical. Since the line transmission capacity is small, all producers correctly anticipate the network to be congested under both the simultaneous-move and sequential-move settings. This leads to the two models predicting the same market outcome at NE. On the other hand, when $c \geq 1$, the price of anarchy under the simultaneous-move formulation is substantially larger than that of the sequential-move formulation. Although the network is uncongested at NE, each producer considers the nodal supply to be fixed under the simultaneous-move model, which essentially serves to limit his strategic influence to the realm of producers located at the his node. This reduces the intensity of competition for each producer under the simultaneous-move model, therefore resulting in a prediction of a larger price of anarchy. Finally, we note that the sequential-move game may have an empty set of NE for intermediate values of c. Such an observation is consistent with existing results indicating the potential for non-existence of NE in networked Stackelberg games [21], [24], [30].

Additionally, the price of anarchy of the simultaneous-move game is increasing in the line capacity c, which reveals the Braess paradox predicted by Theorem 2. In order to elucidate as to why such a Braess paradox arises, we plot the production cost at the unique simultaneous-move NE and the efficient production cost versus the line capacity c in Figures 1c-1d. For $\beta = 1.15$, the aggregate production cost at the NE initially increases in c. Such an paradoxical behavior can be interpreted according to the following arguments. Since the number of producers at node 1 is substantially smaller than the number at node 2, the intensity of competition at node 1 is substantially less than that at node 2. As a result, the nodal price markup above producers' true marginal cost at node 1 is much larger than that at node 2, which is revealed in Figure 1e for small values of the line capacity c. As one increases

[2] We refer the reader to Xu et al. [6] for the definition of this supply function game and the characterization of its unique NE.

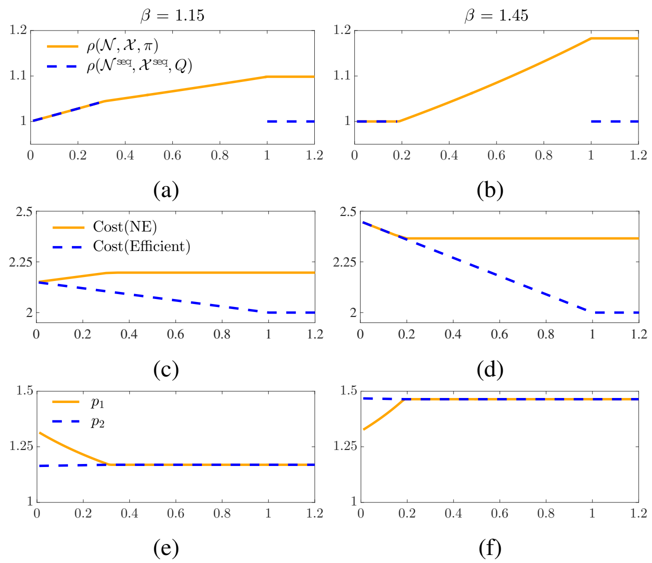

Fig. 1: We fix two different values of β, and vary the transmission line capacity c from 0 to 1.2. Figures 1a-1b plot the price of anarchy under the simultaneous-move and sequential-move formulations. Figures 1c-1d plot the aggregate production cost at an efficient production profile and at the unique NE of the simultaneous-move game. Figures 1e-1f plot the nodal prices at the unique NE of the simultaneous-move game.

c, the production quantity of the expensive producers at node 2 increases, which increases the aggregate production cost at the NE.

V. Conclusion

We conclude with a discussion on two interesting directions for future research. First, our simultaneous-move formulation of the supply function game does not capture the sequential nature of the interaction between the ISO and producers, and, therefore, may not provide an accurate prediction of the market outcome when the network transmission capacity is sufficiently large. Thus, it would be of interest to construct models of competition for electricity markets that capture both the bounded rationality of producers, and the sequential nature of their interaction with the ISO. Additionally, all our analysis of the supply function game amounts to a static equilibrium analysis. As to whether these equilibria can be attained through producers' natural learning dynamics remains unknown.

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