

Stochastic AC Optimal Power Flow with Affine Recourse

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Abstract—With the increasing penetration of intermittent renewable energy sources into the electric power grid, there is an emerging need to develop stochastic optimization methods to enable the reliable and efficient operation of power systems having a large fraction of their power supplied from uncertain resources. In this paper, we formulate the stochastic AC optimal power flow (OPF) problem as a two-stage stochastic program with robust constraints. This problem amounts to an infinite-dimensional nonconvex optimization problem. We develop a finite-dimensional inner approximation as a semidefinite program. Its solution yields an affine recourse policy that is guaranteed to be feasible for the stochastic AC OPF problem.

Index Terms—Optimal power flow, stochastic optimization, affine recourse.

I. INTRODUCTION

The optimal power flow (OPF) problem is a fundamental decision problem in power system operations [1]. In its most general form, OPF is a nonconvex optimization problem, where the objective is to minimize the total cost of generation subject to power balance constraints described by Kirchhoff’s current and voltage laws, and operational constraints reflecting real and reactive limits on power generation, branch flows, and bus voltage magnitudes. The nonconvexity of the OPF problem derives in part from the need to enforce quadratic constraints, which are indefinite in the vector of bus voltages. The treatment of such nonconvexities in the OPF problem has traditionally relied on the use of local methods for constrained optimization, or the use of approximate linear models of power flow to convexify the underlying optimization problem (e.g., DC-OPF). More recently, considerable effort has been made to derive conditions under which an optimal solution to the OPF problem can be obtained through a solution of its semidefinite programming relaxation [2], [3].

Growing public concern surrounding climate change has led many U.S. states to adopt legislation mandating that a significant percentage of their electricity be generated by clean renewable sources. A basic challenge facing the large-scale integration of wind and solar resources derives from the need to compensate the attending uncertainty in their supply of power. At the heart of this challenge is the OPF problem, and the need to develop stochastic optimization methods to enable the reliable and cost-effective operation of power systems having a large fraction of their power supplied from uncertain renewable sources. In its most basic

form, the stochastic OPF problem amounts to a two-stage stochastic program in which the system operator must determine a day-ahead generation schedule, which minimizes the expected cost of generation, given the opportunity for recourse in real-time when the uncertain system variables have been realized. Essentially, the stochastic OPF problem is an infinite-dimensional nonconvex optimization problem. To address the nonconvexity in stochastic OPF, the vast majority of the literature on the topic employs a DC linear approximation of the AC power flow model [4]–[12]. In addition, the majority of the literature relies on affine or piecewise affine approximations of the infinite-dimensional recourse policy space [4]–[13]. And the primary approach to the treatment of uncertainty has focused on either robust [4]–[6], [10], [14] or chance-constrained [7]–[9], [11], [13] formulations.

Summary of Results: In this paper, we formulate a stochastic OPF problem as a two-stage stochastic program with robust constraints. Our primary point of departure from the existing literature is our treatment of the full AC power flow model. To the best of our knowledge, the only papers that treat this model are [13]–[15]. A critical assumption made in these papers is the assumption of exactness of the convex (semidefinite or second-order cone) programming relaxations on which they rely. Exactness of such convex relaxations for the stochastic OPF problem (with recourse) is at present not well understood. Alternatively, in this paper, we explore the extent to which the stochastic OPF problem might be approximated from within by a convex (semidefinite) program.

Faced with an infinite-dimensional nonconvex program with an infinite number of constraints, we develop in Sections IV-A and IV-B a finite-dimensional inner approximation as a semidefinite program. We first restrict the space of admissible recourse policies to be affine in the uncertain system variables. This restriction yields a finite-dimensional nonconvex program subject to an infinite number of constraints, which we further approximate using weak duality to obtain a sufficient set of finitely many nonconvex matrix inequalities. Finally, we address the nonconvexity by employing a convex majorization technique. With this approximation scheme in hand, we provide in Section IV-C a sufficient condition guaranteeing that the inner approximation has a nonempty feasible region. Finally, in Section V, we provide an iterative method, which yields a sequence of affine recourse policies whose costs are nonincreasing. We omit all proofs in this version of the paper due to space constraints.

Notation: Let \mathbf{N} , \mathbf{C} , and \mathbf{R} be the sets of natural, complex, and real numbers, respectively. For any $m \in \mathbf{N}$,

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let $[m] := \{1, 2, \dots, m\}$. Denote by e_i the real i^{th} standard basis vector, of dimension appropriate to the context in which it is used. For any $z_1, z_2 \in \mathbf{C}$, we define a partial ordering on \mathbf{C} by $z_1 \leq z_2$ if and only if $\text{Re}\{z_1\} \leq \text{Re}\{z_2\}$ and $\text{Im}\{z_1\} \leq \text{Im}\{z_2\}$. For a matrix $X \in \mathbf{C}^{m \times n}$, let $[X]_{ij}$ denote its (i, j) entry, and X^* its conjugate transpose. In addition, let $\mathbf{H}^n \subset \mathbf{C}^{n \times n}$ be the space of n -by- n Hermitian matrices. For a matrix $X \in \mathbf{H}^n$, the notation $X \succeq 0$ means that X is positive semidefinite. For a matrix $X \in \mathbf{R}^{m \times n}$, $X \geq 0$ means that X is nonnegative entrywise. Denote by $\text{tr}(X)$ the trace of a matrix $X \in \mathbf{C}^{n \times n}$.

II. SYSTEM MODEL

We begin with a development of a general model for AC optimal power flow under uncertainty. The system we consider consists of a heterogeneous mix of generation and load resources, which differ in terms of their inherent controllability and predictability. The perspective we adopt is that of the system operator (SO), whose objective is to determine the dispatch of available resources in order to minimize the expected cost of meeting demand, while ensuring that all operational limits of generation and transmission facilities are met.¹ The optimization model we consider consists of two stages: day-ahead (DA) and real-time (RT). In day-ahead, the SO must schedule an initial dispatch of its resources subject to uncertainty in the eventual realization of certain system variables, including demand and generation levels of renewable resources. Such DA scheduling decisions are essential, as certain generation resources (e.g., coal and nuclear) have limited ramping capability. In real-time, all uncertain variables are realized, and the SO is provided a recourse opportunity to adjust its DA dispatch schedule in order to balance the system at minimum cost. Ultimately, the determination of a DA schedule, which minimizes the expected cost of dispatch given optimal recourse in real-time, amounts to the solution of a two-stage stochastic programming problem. We formally define this problem in (7).

A. Power Flow Model

We consider an electric power network whose topology is described by a simple graph $\mathcal{G} = ([n], \mathcal{E})$, where the vertex set $[n]$ represents the collection of transmission buses and the edge set \mathcal{E} represents the collection of transmission lines connecting buses. We describe the AC power balance equations according to Kirchhoff's current and voltage laws, which govern the relationship between complex bus voltages and power injections [16]. Let $Y \in \mathbf{C}^{n \times n}$ be the network admittance matrix, $v \in \mathbf{C}^n$ the vector of bus voltages, and $s \in \mathbf{C}^n$ the vector of bus power injections. The AC power balance equations can be expressed as

$$s_i = v^* S_i v, \quad i \in [n], \quad (1)$$

¹This problem is commonly referred to as the *security constrained economic dispatch* (SCED) problem.

where $S_i := Y^* e_i e_i^*$. The complex power flow from bus i to bus j , denoted by $s_{ij} \in \mathbf{C}$, is given by

$$s_{ij} = v^* S_{ij} v, \quad (i, j) \in \mathcal{E}, \quad (2)$$

where $S_{ij} := e_i e_i^* (\hat{y}_{ij}/2 - [Y]_{ij})^* + e_j e_j^* [Y]_{ij}^*$ and $\hat{y}_{ij} \in \mathbf{C}$ denotes the total shunt admittance of line (i, j) . We enforce two classes of constraints. The first requires that bus voltage magnitudes satisfy

$$v_i^{\min} \leq |v_i| \leq v_i^{\max}, \quad i \in [n],$$

where $v_i^{\min}, v_i^{\max} \in \mathbf{R}$ denote upper and lower bounds on the voltage magnitude at bus $i \in [n]$. The second class of constraints enforce line flow capacities. Namely, the real power flow from bus i to bus j must satisfy

$$-\ell_{ij}^{\max} \leq v^* P_{ij} v \leq \ell_{ij}^{\max}, \quad (i, j) \in \mathcal{E},$$

where $P_{ij} := (S_{ij} + S_{ij}^*)/2$ and $\ell_{ij}^{\max} \in \mathbf{R}$ denotes the real power flow capacity of line (i, j) .

B. Uncertainty Model

Uncertainty is modeled by a probability space $(\mathbf{R}^k, \mathcal{B}(\mathbf{R}^k), \mathbb{P})$. The elements, ξ , of the sample space \mathbf{R}^k represent the uncertainty in power injection and the Borel σ -algebra, $\mathcal{B}(\mathbf{R}^k)$, is the set of events that are assigned probabilities by the probability measure \mathbb{P} . For any $k, n \in \mathbf{N}$, let $\mathcal{L}_{k,n}^2$ be the space of all Borel measurable, square-integrable functions from \mathbf{R}^k to \mathbf{C}^n .² We denote by $\mathbb{E}[\cdot]$ the expectation with respect to \mathbb{P} and define $\Xi \subseteq \mathbf{R}^k$ to be the support of \mathbb{P} . We assume throughout that Ξ is a convex compact subset of \mathbf{R}^k given by

$$\Xi := \{\xi \in \mathbf{R}^k \mid \xi_1 = 1, \xi^* W_j \xi \geq 0, j \in [\ell]\},$$

where each matrix $W_j \in \mathbf{R}^{k \times k}$ has the form

$$W_j := \begin{bmatrix} \omega_j & w_j^* \\ w_j & -\Omega_j^* \Omega_j \end{bmatrix}. \quad (3)$$

Here, $\omega_j \in \mathbf{R}$, $w_j \in \mathbf{R}^{k-1}$, and $\Omega_j \in \mathbf{R}^{q_j \times (k-1)}$, for some $q_j \in \mathbf{N}$. The specification that $\xi_1 = 1$ for all $\xi \in \Xi$ allows us to represent affine functions of $(\xi_2, \dots, \xi_k)^*$ as linear functions of ξ . We further assume that for all $j \in [\ell]$, there exists $\xi \in \Xi$ such that $\xi^* W_j \xi > 0$. This assumption, is needed for the proof of Proposition 2. We denote by

$$M := \mathbb{E}[\xi \xi^*] \in \mathbf{R}^{k \times k},$$

the second-order moment matrix. One can readily show that M is real, symmetric, and positive definite (see e.g., [17]).

Remark 1. The above representation of Ξ allows us to model polyhedral uncertainty sets (by setting $\Omega_j = 0$ for all $j \in [\ell]$) as well as uncertainty sets described by convex quadratic functions of ξ .

²A complex-valued function f on \mathbf{R}^k is said to be Borel measurable if both $\text{Re}\{f\}$ and $\text{Im}\{f\}$ are real-valued Borel measurable.

C. Resource Model

Consumer Model: We consider a consumer model in which the real-time demand for power at node i is deterministic and denoted by $d_i \in \mathbf{C}$.

Producer Model: To maintain clarity of exposition, we assume that there is at most a single producer at each bus $i \in [n]$. We consider a producer model in which the real-time supply of power, as determined by the SO, is allowed to depend on the realization of the uncertain system variables ξ . Accordingly, we let $g_i(\xi) \in \mathbf{C}$ denote the power produced at bus i , where $g_i \in \mathcal{L}_{k,1}^2$ is a (recourse) function to be chosen by the SO for each bus $i \in [n]$. Each generator i incurs a cost, which is assumed to be linear in the real power produced. We explicitly define its production cost as

$$\alpha_i \text{Re}\{g_i(\xi)\},$$

where $\alpha_i \geq 0$ for all $i \in [n]$.

In order to capture the potential for uncertainty in the generation capacity available to each producer in real-time, we require that the power generated by each producer i respects the constraints

$$\underline{g}_i(\xi) \leq g_i(\xi) \leq \bar{g}_i(\xi), \quad i \in [n], \quad (4)$$

where $\underline{g}_i(\xi) \in \mathbf{C}$ and $\bar{g}_i(\xi) \in \mathbf{C}$ denote the minimum and maximum power levels, respectively, that producer i can sustain in real-time. This uncertainty in a generator's available capacity can be used to model unscheduled generator outages, and intermittency in renewable power supply. The realized capacities are assumed to satisfy

$$g_i^{\min} \leq \underline{g}_i(\xi) \leq \bar{g}_i(\xi) \leq g_i^{\max}, \quad i \in [n],$$

where $g_i^{\min}, g_i^{\max} \in \mathbf{C}$ are the nameplate minimum and maximum capacities of producer i , respectively. We denote the corresponding vectors by $\underline{g}(\xi), \bar{g}(\xi), g^{\min}, g^{\max} \in \mathbf{C}^n$.

In practice, a generator cannot adjust its production level instantaneously, but rather is limited by a prespecified rate (usually measured in MVA/min) that depends on the type of generator. We reflect generator i 's limited ramping capability with constraints of the form

$$r_i^{\min} \leq g_i(\xi) - g_i^0 \leq r_i^{\max}, \quad i \in [n], \quad (5)$$

where $r_i^{\min}, r_i^{\max} \in \mathbf{C}$ denote generator i 's ramp-down and ramp-up limits, respectively. Here, $g_i^0 \in \mathbf{C}$ denotes the operating point of generator i selected by the SO day-ahead. It is required to satisfy

$$g_i^{\min} \leq g_i^0 \leq g_i^{\max}, \quad i \in [n]. \quad (6)$$

In line with [18], our producer model is general enough to capture a wide range of generator types. We specify several important examples in the following discussion. Given a day-ahead dispatch level g_i^0 that satisfies (6), generator i is said to be:

- *Completely inflexible* (e.g., nuclear), if its allowable range of real-time outputs is given by

$$g_i(\xi) = g_i^0.$$

- *Completely flexible* (e.g., oil, gas), if its allowable range of real-time outputs is given by

$$g_i^{\min} \leq g_i(\xi) \leq g_i^{\max}.$$

- *Intermittent* (e.g., wind, solar), if its allowable range of real-time outputs is given by

$$\underline{g}_i(\xi) \leq g_i(\xi) \leq \bar{g}_i(\xi).$$

We make the following assumption, which is assumed to hold throughout the paper.

Assumption 1. Assume that $g(\xi) = \underline{G}\xi$, and $\bar{g}(\xi) = \bar{G}\xi$, for some matrices $\underline{G}, \bar{G} \in \mathbf{C}^{n \times k}$.

III. STOCHASTIC AC OPTIMAL POWER FLOW

Leveraging on the preceding development, we formulate the two-stage stochastic OPF problem, which has the form

$$\text{minimize } \mathbb{E} \left[\sum_{i=1}^n \alpha_i \text{Re}\{g_i(\xi)\} \right] \quad (7)$$

$$\text{subject to } g_i^0 \in \mathbf{C}, \quad g_i \in \mathcal{L}_{k,1}^2, \quad v \in \mathcal{L}_{k,n}^2, \quad \forall i \in [n],$$

$$g_i^{\min} \leq g_i^0 \leq g_i^{\max}, \quad \forall i \in [n],$$

$$\left. \begin{aligned} \underline{g}_i(\xi) &\leq g_i(\xi) \leq \bar{g}_i(\xi) \\ r_i^{\min} &\leq g_i(\xi) - g_i^0 \leq r_i^{\max}, \\ g_i(\xi) - v(\xi)^* S_i v(\xi) &= d_i, \\ v_i^{\min} &\leq |v_i(\xi)| \leq v_i^{\max}, \\ |v(\xi)^* P_{ij} v(\xi)| &\leq \ell_{ij}^{\max}, \quad \forall (i, j) \in \mathcal{E}, \quad \xi \in \Xi, \end{aligned} \right\} \quad \begin{aligned} &\forall i \in [n], \\ &\forall \xi \in \Xi, \end{aligned}$$

where g_i is a (recourse) function specifying the power generated in real-time by producer $i \in [n]$ and v is a (recourse) function specifying the real-time bus voltage phasors.

Remark 2. In (7), it is possible to include an additional term to the objective function corresponding to day-ahead generation costs. These can be thought of as costs incurred by energy producers for operating their units at a no-load point.

Using the power balance equality constraints to eliminate the decision variables $g_i \in \mathcal{L}_{k,1}^2$, $i \in [n]$ and writing each inequality over the complex numbers as a set of two inequalities over the real numbers, (7) can be posed as an instance of the following class of two-stage stochastic problems

$$\text{minimize } \mathbb{E}[v(\xi)^* A_0 v(\xi)] \quad (\mathcal{P}_r)$$

$$\text{subject to } u \in \mathbf{R}^p, \quad v \in \mathcal{L}_{k,n}^2$$

$$v(\xi)^* A_i v(\xi) + b_i^* u \leq c_i^* \xi, \quad \forall i \in [m], \quad \xi \in \Xi,$$

$$Eu \leq f,$$

where $A_0, A_i \in \mathbf{H}^n$, $b_i \in \mathbf{R}^p$, $c_i \in \mathbf{R}^k$, for all $i \in [m]$, $E \in \mathbf{R}^{s \times p}$, and $f \in \mathbf{R}^s$, for some $m, n, p, s \in \mathbf{N}$. One

can verify that for problem (7), $m = 10n + 2|\mathcal{E}|$, $p = 2n$, and $s = 4n$. The two-stage stochastic problem \mathcal{P}_r amounts to an infinite-dimensional nonconvex optimization problem with an infinite number of constraints. The nonconvexity arises because the feasible set has a non-convex quadratic dependency on the set of complex bus voltages, while the infinite number of constraints derives from the continuous structure of the uncertainty set Ξ .

IV. CONVEX INNER APPROXIMATION

In this Section, we develop a finite-dimensional inner approximation to the two-stage stochastic program \mathcal{P}_r as a semidefinite program. We first restrict the space of admissible recourse policies to be affine in uncertain system variables. Under this restriction, the two-stage stochastic program \mathcal{P}_r becomes a finite-dimensional nonconvex program subject to an infinite number of constraints. Using weak duality, we derive a sufficient set of finitely many nonconvex quadratic matrix inequalities. Lastly, we apply a convex majorization technique, which approximates the nonconvex matrix inequalities by a sufficient set of linear matrix inequalities. We provide in Section IV-C a sufficient condition, which guarantees that the resulting inner approximation has a nonempty feasible region.

A. Affine Recourse

To obtain a tractable inner approximation to the two-stage stochastic program \mathcal{P}_r , we first restrict the functional form of the admissible recourse policies to be linear in the uncertain variables ξ . More precisely, we require that

$$v(\xi) = V\xi \quad (8)$$

for some matrix $V \in \mathbf{C}^{n \times k}$. Given this restriction, we obtain the following semi-infinite nonconvex program whose optimal value stands as an upper bound on the optimal value of the two-stage stochastic program \mathcal{P}_r

$$\text{minimize} \quad \text{tr}(MV^*A_0V) \quad (\mathcal{P}_a)$$

$$\text{subject to} \quad u \in \mathbf{R}^p, V \in \mathbf{C}^{n \times k}$$

$$\xi^*(V^*A_iV)\xi + b_i^*u \leq c_i^*\xi, \quad \forall i \in [m], \xi \in \Xi,$$

$$Eu \leq f.$$

The restriction to affine recourse policies results in an optimization problem \mathcal{P}_a whose decision variables range over finite-dimensional spaces. However, due to the continuous structure of the uncertainty set Ξ , problem \mathcal{P}_a has infinitely many constraints and is, in general, intractable. To account for this, we apply weak duality to obtain a sufficient set of finitely many constraints. We remark that this approximation of the feasible set can also be derived by employing the S -procedure [19]. We have the following Lemma, which follows directly from Proposition 6 in [17].

Lemma 1. Fix $P \in \mathbf{H}^k$, $q \in \mathbf{R}^k$, and $r \in \mathbf{R}$ and let $Q = (e_1q^* + qe_1^*)/2$. Consider the following two statements:

$$(i) \quad \xi^*P\xi + q^*\xi + r \leq 0, \quad \forall \xi \in \Xi,$$

$$(ii) \quad \exists \lambda \in \mathbf{R}^\ell \text{ with } \lambda \leq 0 \text{ and } P+Q+r e_1 e_1^* - \sum_{j=1}^{\ell} \lambda_j W_j \preceq 0.$$

For any $\ell \in \mathbf{N}$, (ii) implies (i). Moreover, if $\ell = 1$, (i) and (ii) are equivalent.

With Lemma 1 in hand, we can replace the infinite constraint set in \mathcal{P}_a with a sufficient set of matrix inequality constraints in the variables $V \in \mathbf{C}^{n \times k}$ and $\lambda_t \in \mathbf{R}^\ell$ for $t \in [m]$. More precisely, let λ_t^* be the t^{th} row of a matrix $\Lambda \in \mathbf{R}^{m \times \ell}$ and for each $i \in [m]$ define matrices

$$C_i := (e_1 c_i^* + c_i e_1^*)/2.$$

The following finite-dimensional nonconvex program \mathcal{P}_f is an inner approximation to both problems \mathcal{P}_a and \mathcal{P}_r .

$$\text{minimize} \quad \text{tr}(MV^*A_0V) \quad (\mathcal{P}_f)$$

$$\text{subject to} \quad u \in \mathbf{R}^p, V \in \mathbf{C}^{n \times k}, \Lambda \in \mathbf{R}^{m \times \ell}$$

$$V^*A_iV - C_i + (b_i^*u)e_1 e_1^* - \sum_{j=1}^{\ell} [\Lambda]_{ij} W_j \preceq 0,$$

$$Eu \leq f,$$

$$\Lambda \leq 0,$$

for all $i \in [m]$.

Remark 3. \mathcal{P}_f is equivalent to \mathcal{P}_a whenever $\ell = 1$.

B. Convexification

In this section, we address the nonconvexity inherent to the quadratic functions in \mathcal{P}_f . We propose a simple method that replaces these functions by a class of convex quadratic functions majorizing (in the positive semidefinite sense) the nonconvex quadratic functions. The proposed method is based on the observation that each nonconvex quadratic function can be expressed as a sum of a convex quadratic function and a concave quadratic function. This is done by decomposing each matrix A_i into its positive semidefinite and negative semidefinite parts. We then approximate the concave function from above with its linearization at a point. Finally, the sum of this linearization with the convex component of the original function yields a convex global overestimator of the original function. The proposed method is illustrated in Figure 1.

More precisely, for each $i \in [m]$, let $A_i = A_i^+ + A_i^-$, where $A_i^+, A_i^- \in \mathbf{H}^n$, denote the positive semidefinite and negative semidefinite parts of A_i , respectively. And let $H_i : \mathbf{C}^{n \times k} \times \mathbf{C}^{n \times k} \rightarrow \mathbf{H}^k$ be a function given by

$$H_i(X, Z) := X^* A_i^+ X + Z^* A_i^- X + X^* A_i^- Z - Z^* A_i^- Z.$$

Clearly, for any given Z , the function $H_i(X, Z)$ is matrix convex in X .³ The following Lemma highlights two properties of H_i , which are used in the proof of Proposition 1 below, and in the proof of Proposition 3 in Section V.

³A function $f : \mathbf{C}^{n \times k} \rightarrow \mathbf{H}^k$ is said to be matrix convex if for all X, Z and $0 \leq \theta \leq 1$, we have $f(\theta X + (1-\theta)Z) \preceq \theta f(X) + (1-\theta)f(Z)$.

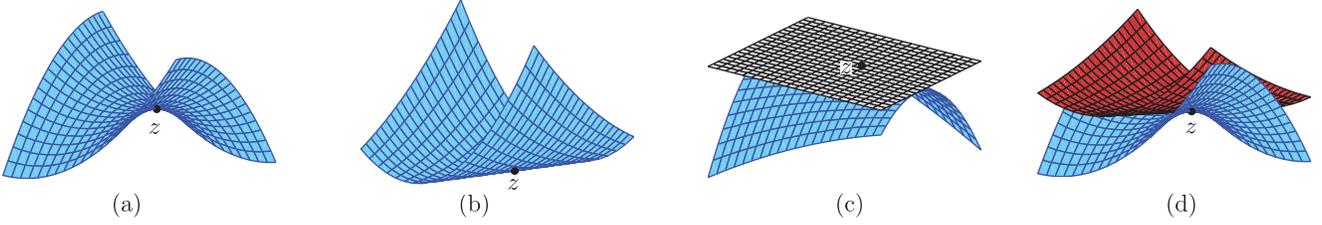


Fig. 1: A nonconvex quadratic function (a) is decomposed as a sum of a convex quadratic function (b) and a concave quadratic function (c). The sum of the convex function (b) and the first order Taylor expansion of the concave function (c) around a given point z , yield a convex quadratic function in (d) majorizing the original nonconvex quadratic.

Lemma 2. Let $Z \in \mathbf{C}^{n \times k}$ be given. For all $i = 0, 1, \dots, m$,

- (i) $X^* A_i X \preceq H_i(X, Z)$, $\forall X \in \mathbf{C}^{n \times k}$,
- (ii) $\text{tr}(M X^* A_i X) \leq \text{tr}(M H_i(X, Z))$, $\forall X \in \mathbf{C}^{n \times k}$.

In light of Lemma 2, we provide in Proposition 1 a semidefinite program whose optimal solution can be mapped to a feasible solution of \mathcal{P}_r . Its proof is a direct consequence of Lemma 2 and we omit it for brevity.

Proposition 1. Let $V_0 \in \mathbf{C}^{n \times k}$ be a given matrix and suppose that $(\bar{u}, \bar{V}, \bar{\Lambda})$ is an optimal solution of the following convex optimization problem

$$\begin{aligned}
 & \text{minimize} && \text{tr}(M H_0(V, V_0)) && (\mathcal{P}_c(V_0)) \\
 & \text{subject to} && u \in \mathbf{R}^p, V \in \mathbf{C}^{n \times k}, \Lambda \in \mathbf{R}^{m \times \ell} \\
 & && H_i(V, V_0) - C_i + (b_i^* u) e_1 e_1^* - \sum_{j=1}^{\ell} [\Lambda]_{ij} W_j \preceq 0, \\
 & && E u \leq f, \\
 & && \Lambda \leq 0,
 \end{aligned}$$

for all $i \in [m]$. Then (\bar{u}, \bar{v}) , where $\bar{v} \in \mathcal{L}_{k,n}^2$ is given by $\bar{v}(\xi) = \bar{V} \xi$ is a feasible solution to the two-stage stochastic program \mathcal{P}_r .

We remark that both the objective function and the matrix inequalities in $\mathcal{P}_c(V_0)$ admit equivalent reformulations as linear matrix inequalities by invoking Schur's complement.

C. Guarantees on Feasibility

A major challenge to the implementation of the convexification technique developed in Section IV-B lies in computing a matrix $V_0 \in \mathbf{C}^{n \times k}$, which yields a nonempty feasible set for the finite-dimensional convex inner approximation $\mathcal{P}_c(V_0)$. In this Section, we provide a method for computing one such matrix. This method entails the computation of a feasible generation schedule without recourse – in other words, an open loop solution to the two-stage stochastic program \mathcal{P}_r . To do so, one needs to first characterize the guaranteed range of available power supply at each bus in the network. More precisely, we let $[\gamma_i^{\min}, \gamma_i^{\max}]$, denote the guaranteed range of power supply at bus i , where

$$\gamma_i^{\min} := \max_{\xi \in \Xi} \underline{g}_i(\xi), \quad \gamma_i^{\max} := \min_{\xi \in \Xi} \bar{g}_i(\xi).$$

Due to the structure of the uncertainty set Ξ and Assumption 1, each of these quantities can be computed in polynomial time by solving a second-order cone program. An open loop solution to the two-stage stochastic program can then be computed by solving the following deterministic OPF problem

$$\text{minimize} \quad v^* \left(\sum_{i=1}^n \alpha_i \frac{S_i + S_i^*}{2} \right) v \quad (9)$$

subject to $v \in \mathbf{C}^n$

$$\begin{aligned}
 & \gamma_i^{\min} - d_i \leq v^* S_i v \leq \gamma_i^{\max} - d_i, \quad \forall i \in [n], \\
 & v_i^{\min} \leq |v_i| \leq v_i^{\max}, \quad \forall i \in [n], \\
 & -\ell_{ij}^{\max} \leq v^* P_{ij} v \leq \ell_{ij}^{\max}, \quad \forall (i, j) \in \mathcal{E}.
 \end{aligned}$$

Although deterministic, the above optimization problem is non-convex and NP-hard, in general. There are, however, many off-the-shelf optimization routines (e.g., MATPOWER [20]), which are reliable in their ability to obtain locally optimal solutions to problem (9). The following Proposition shows that a feasible solution to problem (9) can be used to construct a matrix V_0 , which yields a nonempty feasible region for the finite-dimensional convex inner approximation of the two-stage stochastic program \mathcal{P}_r .

Proposition 2. Let v_0 be an optimal solution to (9), and define $V_0 = v_0 e_1^*$. Then, the optimization problem $\mathcal{P}_c(V_0)$ has a nonempty feasible region.

V. SUCCESSIVE CONVEX APPROXIMATIONS

Given an optimal solution v_0 to the deterministic OPF problem (9), define the matrix $V_0 := v_0 e_1^*$ and consider an iterative algorithm of the form

$$(u_{t+1}, V_{t+1}, \Lambda_{t+1}) \in \underset{(u, V, \Lambda) \in \mathcal{F}(V_t)}{\text{argmin}} \text{tr}(M H_0(V, V_t)). \quad (10)$$

Here, $\mathcal{F}(V_t)$ denotes the feasible set of problem $\mathcal{P}_c(V_t)$. The algorithm (10) can be interpreted as implementing a successive convex majorization-minimization method. We have the following Proposition, which highlights some of its properties.

Proposition 3. For any iteration $t \geq 0$ of (10), the following properties hold.

(i) $\mathcal{F}(V_t)$ is nonempty.

(ii) Any solution $(u_{t+1}, V_{t+1}, \Lambda_{t+1})$ satisfies

$$\text{tr}(MV_{t+1}^* A_0 V_{t+1}) \leq \text{tr}(MV_t^* A_0 V_t).$$

Proposition 3 shows that the iterative algorithm (10) yields a sequence of feasible solutions to the two-stage stochastic program \mathcal{P}_r with nonincreasing costs.

VI. CONCLUSION

We formulate the stochastic AC optimal power flow (OPF) problem as a two-stage stochastic program with robust constraints. We provide a method to approximate the stochastic AC OPF problem from within by a semidefinite program. Its solution yields an affine recourse policy in the uncertain system variables, which is guaranteed to be feasible for the stochastic AC OPF problem. We also develop an iterative method, which yields a sequence of feasible affine recourse policies whose costs are nonincreasing. Moving forward, we plan to develop a convex outer approximation of the stochastic AC OPF problem to bound the suboptimality incurred by the feasible solutions generated according the method developed in this paper.

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