A Hierarchy of Polyhedral Approximations of Robust Semidefinite Programs

Raphael Louca Eilyan Bitar

***Abstract*—Robust semidefinite programs are NP-hard in general. In contrast, robust linear programs admit equivalent reformulations as finite-dimensional convex programs provided that the problem data are parameterized affinely in the uncertain parameters; and that the underlying uncertainty set is described by an affine slice of a proper cone. In this paper, we propose a hierarchy of inner and outer polyhedral approximations to the positive semidefinite (PSD) cone that are exact in the limit. We apply these polyhedral approximations to the PSD cone to obtain a computationally tractable hierarchy of inner and outer approximations to the robust semidefinite program, which are similarly exact in the limit. We investigate the strengths and limitations of the proposed approach with a detailed numerical study.**

***Index Terms*—Robust semidefinite programs.**

I. Introduction

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space. A conic linear program is an optimization problem of the form

$$\begin{array}{ll} \underset{x\in\mathbf{R}^m}{\text{minimize}} & c^\top x \\ \text{subject to} & \mathcal{A}(x) \in K, \end{array}$$

where the affine map $\mathcal{A} : \mathbf{R}^m \to \mathbf{R}^n$ and the vector $c \in \mathbf{R}^m$ are the given problem data and $K \subseteq \mathbf{R}^n$ is a closed convex cone. This class of problems includes as special cases, linear programs, second-order cone programs, and semidefinite programs (SDPs). A plethora of practical problems in engineering and applied mathematics can be formulated as conic linear programs [1]–[3]. Of practical interest is the setting in which the affine map $\mathcal{A}$ is not exactly known.[1] Rather, what is known is that $\mathcal{A}$ lies in a set U, which is assumed to be a convex compact subset of $L(\mathbf{R}^m, \mathbf{R}^n)$, the space of all affine maps from $\mathbf{R}^m$ to $\mathbf{R}^n$. A robust solution is one which satisfies the constraints for every possible realization of the uncertain data; that is, a vector $x \in \mathbf{R}^m$ satisfying $\mathcal{A}(x) \in K$, for all $\mathcal{A} \in U$. The corresponding *robust conic linear program* [4], [5] is defined as

$$\begin{array}{lll} \underset{x\in\mathbf{R}^m}{\text{minimize}} & c^\top x & \\ \text{subject to} & \mathcal{A}(x) \in K, & \forall\, \mathcal{A} \in U. \end{array}$$

The semi-infinite[2] structure of the robust program renders it computationally intractable in general. There are, however, certain cones K and uncertainty sets U for which the robust program admits an equivalent reformulation as a finite-dimensional convex optimization problem [4], [6]. For example, consider an uncertain linear program (i.e., $K = \mathbf{R}^n_+$) in which the mapping $\mathcal{A}$ is affinely parameterized in the uncertain parameters. If the uncertainty set is either polytopic, ellipsoidal, or semidefinite representable, then the robust program can be reformulated as a linear, second-order cone, or semidefinite program, respectively. Robust second-order cone and semidefinite programs do not, in general, admit similar tractable reformulations under the same conic representations of the uncertainty set. There are, however, certain characterizations of the uncertainty set for which robust second-order cone and semidefinite programs admit computationally tractable reformulations or inner approximations [6], [7]. We discuss such results in more detail in Section II-B.

Summary of Results: Let $\mathbf{S}^n$ be the space of $n \times n$ real symmetric matrices and $\mathbf{S}^n_+$ the subset of positive semidefinite (PSD) matrices. In this paper, we study robust semidefinite programs with uncertainty sets of the form

$$U := \left\{ \sum_{i=1}^k \xi_i \mathcal{A}_i \;\middle|\; \xi \in \Xi \right\}, \tag{1}$$

where the parameter uncertainty set $\Xi \subseteq \mathbf{R}^k$ is assumed to be a compact convex set described as an affine slice of a proper cone. The robust semidefinite program has the form

$$\begin{array}{ll} \underset{x\in\mathbf{R}^m}{\text{minimize}} & c^\top x \\ \text{subject to} & \displaystyle\sum_{i=1}^k \xi_i \mathcal{A}_i(x) \;\in\; \mathbf{S}^n_+, \quad \forall\, \xi \in \Xi, \end{array} \tag{$\mathcal{P}$}$$

where $c \in \mathbf{R}^m$ and $\mathcal{A}_1, \ldots, \mathcal{A}_k \in L(\mathbf{R}^m, \mathbf{S}^n)$.

In this paper, we propose an approximation scheme, which gives rise to a hierarchy of inner and outer approximations to the robust semidefinite program. The crux of our approach centers on the construction of a hierarchy of inner and outer polyhedral approximations to the positive semidefinite cone, which are exact in the limit.[3] At each level of the hierarchy, we apply the inner (outer) polyhedral approximation

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[1] Without loss of generality, we assume that $c \in \mathbf{R}^n$ is fixed and known.

[2] A semi-infinite program is an optimization problem involving finitely many decision variables, but an infinite number of constraints.

[3] Contrary to the second-order cone [8], it is shown in [9] that it is, in general, not possible to approximate the positive semidefinite cone to within an arbitrary accuracy with a polynomial number of linear inequalities.

to the positive semidefinite cone to obtain a robust linear program that is an inner (outer) approximation to the robust semidefinite program. The resulting robust linear programs admit equivalent reformulations as finite-dimensional convex programs [4]. At each level of the hierarchy, any solution to the inner robust linear program will be feasible for the robust semidefinite program. Moreover, the optimal value of the outer robust linear program serves as a lower bound on the optimal value of the robust semidefinite program – thereby providing a bound on the suboptimality of the feasible point generated by the inner robust linear program.

The remainder of this paper is organized as follows. In Section II, we present existing results prertaining to exact reformulations and inner approximations of robust SDPs. Section III presents a scheme for constructing polyhedral approximations to the positive semidefinite cone. Sections IV-V contain our main results. We evaluate the strengths and limitations of the proposed approach with a detailed numerical study in Section VI. We conclude the paper with directions for future research in Section VII.

Notation: Denote by $\mathbf{N}$ the set of natural numbers. Given any positive integer $n \in \mathbf{N}$, define the set $[n] := \{1, 2, \ldots, n\}$. Let e_i be the i^{th} real standard basis vector, of dimension appropriate to the context in which it is used. The dual cone of a given cone $K \subseteq \mathbf{R}^n$ is defined as $K^* = \{y \in \mathbf{R}^n \mid x^\top y \geq 0, \ \forall\, x \in K\}$.

II. Preliminaries

A. Uncertainty Model

The uncertainty set Ξ is assumed to be a convex compact subset of $\mathbf{R}^k$ given by

$$\Xi \ := \ \{\xi \in \mathbf{R}^k \mid \xi_1 = 1, \ B_j \xi \in C_j, \ j \in [d]\},$$

where $C_j \subseteq \mathbf{R}^k$ is a proper cone and $B_j \in \mathbf{R}^{k\times k}$ for all $j \in [d]$. The requirement that $\xi_1 = 1$ is for notational convenience, as it enables the representation of affine functions of $(\xi_2, \ldots, \xi_k)^\top$ as linear functions of ξ. We assume that the linear hull of Ξ spans $\mathbf{R}^k$. Such assumption is without loss of generality, and is used in the proof of the following Lemma, which shows that a semi-infinite linear constraint can be replaced by a finite number of linear constraints.

Lemma 1. *Let $z \in \mathbf{R}^k$. Then, the following two statements are equivalent*

(i) $z^\top \xi \geq 0, \ \ \forall\, \xi \in \Xi,$

(ii) $\exists\ \mu \in \mathbf{R}$ *and* $\lambda_j \in \mathbf{R}^k$, *for all* $j \in [d]$ *such that*

$$\mu e_1 + \sum_{j=1}^{d} B_j^\top \lambda_j = z, \ \ \mu \geq 0, \ \ \text{and} \ \ \lambda_j \in C_j^*, \ \forall\, j \in [d].$$

The proof of Lemma 1 is omitted, as it relies on a straightforward argument based on strong duality. See, for example, Theorem 1.3.4 in [6].

B. Exact and Approximate Solutions to Robust SDPs

In this section, we review results from the existing literature, which establishes conditions on the structure of the uncertainty set Ξ that enable the tractable reformulation or inner approximation of the robust semidefinite program $\mathcal{P}$.

First, consider so-called *scenario-generated uncertainty sets* described by

$$\Xi = \text{conv}\ \{\xi^1, \ldots, \xi^N\}, \tag{2}$$

where $\xi^j \in \mathbf{R}^k$ for all $j = 1, \ldots, N$. For such uncertainty sets, the robust semidefinite program admits an equivalent reformulation as a semidefinite program. We have the following Theorem from [6, Chap. 8.1].

Theorem 1. *Consider a scenario-generated uncertainty set Ξ of the form* (2)*. Then, the semidefinite program*

$$\begin{aligned} &\underset{x \in \mathbf{R}^m}{\text{minimize}} && c^\top x \\ &\text{subject to} && \sum_{i=1}^{k} \xi_i^j \mathcal{A}_i(x) \in \mathbf{S}_+^n, \ \ \forall\ j \in [N], \end{aligned} \tag{3}$$

is equivalent to the robust semidefinite program $\mathcal{P}$.

Now, consider *norm-bounded uncertainty sets* described by

$$\begin{aligned}\Xi = \{\xi \in \mathbf{R}^k \mid \xi = (1, \xi^1; \ldots; \xi^N), \ \xi^j \in \mathbf{R}^{n_j}, \\ \|\xi^j\|_2 \leq \rho, \ \forall\ j \in [N]\},\end{aligned} \tag{4}$$

where $\rho \in \mathbf{R}_+$. It is well known that if the norm-bounded uncertainty set is *unstructured* (i.e., $N = 1$), then the robust semidefinite program can be equivalently reformulated as a semidefinite program. Further, if the norm-bounded uncertainty set is *structured* (i.e., $N > 1$), then the robust semidefinite program can be approximated from within by a semidefinite program. We summarize these results in the following Theorem. We refer the reader to [2, Thm. 6.2.1] and [6, Thm. 9.1.2 & Thm. 8.2.3] for their proofs.

Theorem 2. *Consider a norm-bounded uncertainty set Ξ of the form* (4)*. Define $\mu_0 := 1$ and $\mu_j := \sum_{s=1}^{j} n_s$ for each $j \in [N]$. Consider the semidefinite program*

$$\begin{aligned} &\text{minimize} && c^\top x \\ &\text{subject to} && x \in \mathbf{R}^m, \ S_j, \ Q_j \in \mathbf{S}^n, \quad \forall\ j \in [N], \\ &&& F_j(x, S_j, Q_j) \in \mathbf{S}_+^n, \qquad \forall\ j \in [N], \\ &&& 2\mathcal{A}_1(x) - \sum_{j=1}^{N} (S_j + Q_j) \in \mathbf{S}_+^n, \end{aligned} \tag{5}$$

where

$$F_j(x, S_j, Q_j) := \begin{bmatrix} S_j & \rho\mathcal{A}_{\mu_{j-1}+1}(x) & \ldots & \rho\mathcal{A}_{\mu_j}(x) \\ \rho\mathcal{A}_{\mu_{j-1}+1}(x) & Q_j & & \\ \vdots & & \ddots & \vdots \\ \rho\mathcal{A}_{\mu_j}(x) & & & Q_j \end{bmatrix}.$$

It holds that:

(i) The projection of the feasible set of the semidefinite program (5) *onto the space of x variables is contained in the feasible set of robust semidefinite program $\mathcal{P}$.*

(ii) If $N = 1$, then the semidefinite program (5) *is equivalent to the robust semidefinite program $\mathcal{P}$.*

Lastly, for general *semi-algebraic uncertainty sets* Ξ, Scherer and Hol [7] develop a method based on sum of squares optimization to approximate the robust semidefinite program $\mathcal{P}$ from within by a semidefinite program.[4] We refer the reader to Theorem 1 of [7] for the details of their construction.

III. Polyhedral Hierarchies of the PSD Cone

In this section, we propose a method for constructing a hierarchy of polyhedral approximations to the positive semidefinite cone.[5] First, note that the positive semidefinite cone can be expressed as

$$\mathbf{S}_+^n \ = \ \{X \in \mathbf{S}^n \mid v^\top X v \geq 0, \ \forall \ v \in \boldsymbol{\Delta}\}, \tag{6}$$

where $\boldsymbol{\Delta} \coloneqq \{v \in \mathbf{R}^n \mid \|v\|_1 = 1\}$ denotes the boundary of the cross polytope in $\mathbf{R}^n$. An outer approximation of the cone can be constructed by requiring that the quadratic inequalities in (6) hold only for a finite number of points on the boundary of the cross polytope. More formally, for each $r \in \mathbf{N}$, define the finite set

$$\boldsymbol{\Delta}_r \ \coloneqq \ \ \{x \in \boldsymbol{\Delta} \mid 2^r x \in \mathbf{N}^n\}, \tag{7}$$

and the corresponding polyhedral cone

$$O_r^n \ \coloneqq \ \{X \in \mathbf{S}^n \mid v^\top X v \geq 0, \ \forall \ v \in \boldsymbol{\Delta}_r\}. \tag{8}$$

The proposed construction yields a hierarchy of outer polyhedral approximations to the positive semidefinite cone, i.e.,

$$O_0^n \ \supseteq \ O_1^n \ \supseteq \ \cdots \ \supseteq \ \mathbf{S}_+^n. \tag{9}$$

Remark 1 (Ordering the Elements in $\boldsymbol{\Delta}_r$)**.** The definition of the polyhedral cone O_r^n implies a system of $p = (1/2)|\boldsymbol{\Delta}_r|$ linear inequalities. The factor of 1/2 arises because $v \in \boldsymbol{\Delta}_r$ implies that $-v \in \boldsymbol{\Delta}_r$. In the sequel, it will be convenient to order the elements in $\boldsymbol{\Delta}_r$ according to $v_1, \ldots, v_{2p}$, where we require that $v_{p+i} = -v_i$ for all $i \in [p]$.

Taking the dual of each cone in the outer polyhedral hierarchy yields a hierarchy of inner polyhedral approximations to the positive semidefinite cone. Namely, for each level in the hierarchy $r \in \mathbf{N}$, let $p = (1/2)|\boldsymbol{\Delta}_r|$ and define the polyhedral cone I_r^n as the dual cone of O_r^n:

$$I_r^n \ \coloneqq \ (O_r^n)^* = \left\{ \sum_{i=1}^p y_i (v_i v_i^\top) \ \middle| \ y \geq 0 \right\}. \tag{10}$$

As the cone of positive semidefinite matrices is self-dual, it is immediate to establish the nested ordering:

$$I_0^n \ \subseteq \ I_1^n \ \subseteq \ \cdots \ \subseteq \ \mathbf{S}_+^n. \tag{11}$$

We summarize our results thus far in Proposition 1, and establish that both the inner and the outer hierarchies of polyhedral cones converge to the PSD cone asymptotically. The proof is omitted, as it relies on arguments identical to those used to prove Theorems 2.1 and 2.2 in [10].

Proposition 1. *For each level $r \in \mathbf{N}$, it holds that:*

(i) $O_r^n \supseteq O_{r+1}^n \supseteq \mathbf{S}_+^n$, and

$$\bigcap_{\ell \in \mathbf{N}} O_\ell^n \ = \ \mathbf{S}_+^n.$$

(ii) $I_r^n \subseteq \ I_{r+1}^n \subseteq \mathbf{S}_+^n$, and

$$\mathrm{cl}\left(\bigcup_{\ell \in \mathbf{N}} I_\ell^n \right) = \mathbf{S}_+^n,$$

where $\mathrm{cl}(S)$ denotes the closure of a set S.

Although exact in the limit, the number of inequalities required to describe the inner an outer polyhedral cones does not scale gracefully with the level of the hierarchy r. In fact, it is straightforward to show that the number of discretization points, $|\boldsymbol{\Delta}_r|$, used to approximate the semidefinite cone is exponential in the level r.

Remark 2 (Levels 0 & 1 in the Hierarchy)**.** One can show that levels 0 and 1 of the proposed inner polyhedral hierarchy can be identified with the cone of *nonnegative diagonal matrices* and the cone of *diagonally dominant matrices with nonnegative diagonal entries*, respectively. First, notice that $\boldsymbol{\Delta}_0$ and $\boldsymbol{\Delta}_1$ are equal to

$$\begin{aligned} &\boldsymbol{\Delta}_0 = \{\pm e_i \mid i \in [n]\}, \\ &\boldsymbol{\Delta}_1 = \boldsymbol{\Delta}_0 \cup \{\pm(1/2)(e_i \pm e_j) \mid 1 \leq i < j \leq n\}. \end{aligned}$$

It follows from (8) that the corresponding outer polyhedral cones are given by

$$\begin{aligned} O_0^n &= \{X \in \mathbf{S}^n \mid X_{ii} \geq 0, \ i \in [n]\}, \\ O_1^n &= O_0^n \cap \{X \in \mathbf{S}^n \mid X_{ii} + X_{jj} \pm 2X_{ij} \geq 0, \\ &\qquad\qquad\qquad\qquad\qquad 1 \leq i < j \leq n\}. \end{aligned}$$

Taking their dual, we have the inner polyhedral cones described by

$$\begin{aligned} I_0^n &= \{X \in \mathbf{S}^n \mid X_{ii} \geq 0, \ i \in [n]; \ X_{ij} = 0, \ \forall \ i \neq j\}, \\ I_1^n &= \Big\{X \in \mathbf{S}^n \ \Big| \ X_{ii} \geq \sum_{j \neq i} |X_{ij}|, \ i \in [n]\Big\}. \end{aligned}$$

Among other applications, the cone of diagonally dominant matrices with nonnegative diagonal entries has recently been proposed in [11] as a method to approximate semidefinite programs for sum of squares optimization.

We close this section by mentioning a related approach form the literature [12], which develops an adaptive cutting

[4]In fact, their method method applies to the more general setting in which the set U is polynomially parameterized in the uncertain parameter ξ.

[5]The approach we consider is closely related to the construction of polyhedral hierarchies for the copositive cone proposed in [10].

plane method to construct linear programming approximations of semidefinite programs. Although not directly applicable to the setting considered in this paper, it would be of interest to explore the extent to which these techniques might be extended to approximate robust semidefinite programs.

IV. Outer Hierarchy for the Robust SDP

We provide in this section a method for constructing a hierarchy of tractable outer approximations to the robust semidefinite program $\mathcal{P}$. For each level in the hierarchy $r \in \mathbf{N}$, define a robust linear program according to

$$\begin{aligned} \underset{x\in\mathbf{R}^m}{\text{minimize}} \quad & c^\top x \\ \text{subject to} \quad & \sum_{i=1}^k \xi_i \mathcal{A}_i(x) \in O_r^n, \quad \forall\ \xi \in \Xi. \end{aligned} \tag{$\mathcal{P}_r^O$}$$

Since $O_r^n \supseteq \mathbf{S}_+^n$, it follows that the optimal value of the above robust linear program stands as a lower bound on the optimal value of the robust semidefinite program $\mathcal{P}$.

In what follows, we provide a tractable reformulation of the (semi-infinite) robust linear program.[6] First, notice that the semi-infinite constraint in the robust linear program $\mathcal{P}_r^O$ holds if and only if

$$v_j^\top \left(\sum_{i=1}^k \xi_i \mathcal{A}_i(x)\right) v_j \geq 0, \qquad \begin{array}{l} \forall\ v_j \in \boldsymbol{\Delta}_r,\ j \in [p], \\ \forall\ \xi\ \in \Xi. \end{array}$$

This semi-infinite system of inequalities can be expressed more compactly as

$$V(x)\xi \geq 0, \quad \forall\ \xi \in \Xi,$$

where the mapping $V : \mathbf{R}^m \to \mathbf{R}^{p\times k}$ is defined as

$$V(x) := \begin{bmatrix} v_1^\top \mathcal{A}_1(x) v_1 & \dots & v_1^\top \mathcal{A}_k(x) v_1 \\ \vdots & \ddots & \vdots \\ v_p^\top \mathcal{A}_1(x) v_p & \dots & v_p^\top \mathcal{A}_k(x) v_p \end{bmatrix}. \tag{12}$$

A direct application of Lemma 1 yields the finite-dimensional conic linear program in Proposition 2, which is an equivalent reformulation of the robust linear program $\mathcal{P}_r^O$.

Proposition 2. *Fix $r \in \mathbf{N}$ and let $p = (1/2)|\boldsymbol{\Delta}_r|$. The robust linear program $\mathcal{P}_r^O$ admits an equivalent reformulation as the following finite-dimensional conic linear program*

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & x \in \mathbf{R}^m,\ \mu \in \mathbf{R}^p,\ \Lambda_j \in \mathbf{R}^{k\times p}, \quad \forall\ j \in [d] \\ & \mu e_1^\top + \sum_{j=1}^d \Lambda_j^\top B_j - V(x) = 0, \\ & \Lambda_j e_i \in C_j^*, \quad \forall\ (i,j) \in [p]\times[d], \\ & \mu \geq 0. \end{aligned} \tag{13}$$

Let ℓ_r^{opt} denote the optimal value of the above program. It holds that $\ell_r^{opt} \leq \ell_{r+1}^{opt} \leq v^{opt}$ for each hierarchy level $r \in \mathbf{N}$.

[6] Here, we let $p = (1/2)|\boldsymbol{\Delta}_r|$ denote the number of hyperplanes defining the polyhedral cone O_r^n, and suppose that the elements of $\boldsymbol{\Delta}_r$ are ordered as described in Remark 1.

V. Inner Hierarchy for the Robust SDP

We provide in this section two methods for constructing a hierarchy of tractable inner approximations to the robust semidefinite program $\mathcal{P}$. For each level in the hierarchy $r \in \mathbf{N}$, define a robust linear program according to

$$\begin{aligned} \underset{x\in\mathbf{R}^m}{\text{minimize}} \quad & c^\top x \\ \text{subject to} \quad & \sum_{i=1}^k \xi_i \mathcal{A}_i(x) \in I_r^n, \quad \forall\ \xi \in \Xi. \end{aligned} \tag{$\mathcal{P}_r^I$}$$

Since $I_r^n \subseteq \mathbf{S}_+^n$, it follows that any feasible solution to the above robust linear program is guaranteed to be feasible for the robust semidefinite program $\mathcal{P}$. As a result, its optimal value stands as an upper bound on the optimal value of the robust semidefinite program $\mathcal{P}$.

In what follows, we present two methods for solving the robust linear program $\mathcal{P}_r^I$. The first approach rests on an equivalent reformulation of the robust linear program as a finite-dimensional conic program. The second approach relies on a conservative approximation of the robust linear program as a finite-dimensional conic program with fewer variables and constraints than the first approach. Naturally, the resulting gain in computational efficiency is offset by a potential increase in conservatism of solutions obtained.

A. Equivalent Reformulation of $\mathcal{P}_r^I$

The V(ertex)-representation of the polyhedral cone I_r^n in (10) precludes a direct application of Lemma 1 to the semi-infinite constraint in $\mathcal{P}_r^I$. However, the Weyl-Minkowski theorem [13, Theorem 3.2] ensures that the polyhedral cone I_r^n has an equivalent H(yperplane)-representation given by

$$I_r^n = \{X \in \mathbf{S}^n \mid h_j^\top X h_j \geq 0,\ j \in [\ell]\}, \tag{14}$$

for some vectors $h_j \in \mathbf{R}^n$ and some positive integer $\ell \in \mathbf{N}$. We remark that an H-representation of I_r^n can be obtained from its V-represenation through Fourier-Motzkin elimination [3].

Using the H-representation of I_r^n in (14), one can equivalently reformulate the semi-infinite constraint in $\mathcal{P}_r^I$ as

$$H(x)\xi \geq 0, \quad \forall\ \xi \in \Xi,$$

where the mapping $H : \mathbf{R}^m \to \mathbf{R}^{\ell\times k}$ is defined as

$$H(x) := \begin{bmatrix} h_1^\top \mathcal{A}_1(x) h_1 & \dots & h_1^\top \mathcal{A}_k(x) h_1 \\ \vdots & \ddots & \vdots \\ h_\ell^\top \mathcal{A}_1(x) h_\ell & \dots & h_\ell^\top \mathcal{A}_k(x) h_\ell \end{bmatrix}. \tag{15}$$

A direct application of Lemma 1 yields the finite-dimensional conic linear program in Proposition 3, which is an equivalent reformulation of the robust linear program $\mathcal{P}_r^I$.

Proposition 3. *Fix $r \in \mathbf{N}$. The robust linear program $\mathcal{P}_r^I$ admits an equivalent reformulation as the following finite-dimensional conic linear program*

$$\begin{aligned} &\text{minimize} && c^\top x && (16)\\ &\text{subject to} && x \in \mathbf{R}^m,\ \mu \in \mathbf{R}^\ell,\ \Lambda_j \in \mathbf{R}^{k\times\ell}, \quad \forall\, j \in [d] \\ &&& \mu e_1^\top + \sum_{j=1}^d \Lambda_j^\top B_j - H(x) = 0, \\ &&& \Lambda_j e_i \in C_j^*, \quad \forall\, (i,j) \in [\ell] \times [d], \\ &&& \mu \geq 0. \end{aligned}$$

Let u_r^{opt} denote the optimal value of the above program. It holds that $u_r^{opt} \geq u_{r+1}^{opt} \geq v^{opt}$ for each hierarchy level $r \in \mathbf{N}$.

A practical drawback of the approximation provided in Proposition 3 is that it requires a preprocessing step to obtain a H-representation of the polyhedral cone I_r^n from its V-representation. Besides being computationally demanding to compute, an H-representation of the polyhedral cone may require a large number of hyperplanes for its specification. This, in turn, gives rise to a large number of constraints in problem (16), thereby jeopardizing its tractability. This practical limitation raises the question as to whether it is possible to work directly with the vertex representation of the polyhedral cone I_r^n to obtain an inner approximation of the robust linear program $\mathcal{P}_r^I$ that is less computationally demanding to solve. We discuss one such approach in the following section.

B. Inner Approximation of $\mathcal{P}_r^I$

We begin with an equivalent reformulation of the robust linear program $\mathcal{P}_r^I$ that will prove useful in the sequel. Using the V-representation of the polyhedral cone I_r^n, it is straightforward to show that the robust linear program $\mathcal{P}_r^I$ can be exactly reformulated as the infinite-dimensional program

$$\begin{aligned} &\text{minimize} && c^\top x && (17)\\ &\text{subject to} && x \in \mathbf{R}^m,\ \mathsf{y}_j \in \mathcal{L}_k, \quad \forall\, j \in [p] \\ &&& \sum_{i=1}^k \xi_i \mathcal{A}_i(x) = \sum_{j=1}^p \mathsf{y}_j(\xi)(v_j v_j^\top), \quad \forall\, \xi \in \Xi \\ &&& \mathsf{y}_j(\xi) \geq 0, \quad \forall\, j \in [p],\ \xi \in \Xi, \end{aligned}$$

where $\mathcal{L}_k$ is the infinite-dimensional space of all functions from $\mathbf{R}^k$ to $\mathbf{R}$. The above optimization problem is intractable in general. To obtain a tractable inner approximation of (17), we restrict the functional form of the functions y_j ($j \in [p]$) to be linear in the uncertain parameter ξ – an approach to complexity reduction originally proposed by Ben-Tal et al. [14]. More precisely, for each $j \in [p]$, we require that

$$\mathsf{y}_j(\xi) \ = \ y_j^\top \xi,$$

for some vector $y_j \in \mathbf{R}^k$. Such restriction gives rise to the following robust linear program, which amounts to an inner approximation of the original robust linear program $\mathcal{P}_r^I$.

$$\begin{aligned} &\text{minimize} && c^\top x && (\overline{\mathcal{P}}_r^I)\\ &\text{subject to} && x \in \mathbf{R}^m,\ y_j \in \mathbf{R}^k, \quad \forall\, j \in [p] \\ &&& \sum_{i=1}^k \xi_i \Big(\mathcal{A}_i(x) - \sum_{j=1}^p y_j^\top e_i (v_j v_j^\top)\Big) = 0,\ \forall\, \xi \in \Xi \\ &&& y_j^\top \xi \geq 0, \quad \forall\, j \in [p],\ \xi \in \Xi. \end{aligned}$$

We now develop an equivalent reformulation of the (restricted) robust linear program $\overline{\mathcal{P}}_r^I$ as a finite-dimensional conic linear program. First notice that, since the semi-infinite equality constraint in $\overline{\mathcal{P}}_r^I$ must hold for all $\xi \in \Xi$, the linear hull of Ξ must be contained in the nullspace of the linear map

$$\xi \mapsto \sum_{i=1}^k \xi_i \Big(\mathcal{A}_i(x) - \sum_{j=1}^p y_j^\top e_i (v_j v_j^\top)\Big). \qquad (18)$$

Moreover, since Ξ is assumed to span all of $\mathbf{R}^k$, the equality constraint holds if and only if the expression inside the parentheses in (18) is equal to zero for all $i \in [k]$. Finally, a direct application of Lemma 1 to the remaining semi-infinite inequality constraints in $\overline{\mathcal{P}}_r^I$ yields the finite-dimensional conic linear program in Proposition 4, which is an equivalent reformulation of the robust linear program $\overline{\mathcal{P}}_r^I$.

Proposition 4. *Fix $r \in \mathbf{N}$. The robust linear program $\overline{\mathcal{P}}_r^I$ admits an equivalent reformulation as a finite-dimensional conic linear program given by*

$$\begin{aligned} &\text{minimize} && c^\top x && (19)\\ &\text{subject to} && x \in \mathbf{R}^m,\ \mu \in \mathbf{R}^p,\ \Lambda_t \in \mathbf{R}^{k\times p}, \quad \forall\, t \in [d] \\ &&& \mathcal{A}_i(x) = \sum_{j=1}^p e_j^\top \Big(\mu e_1^\top + \sum_{t=1}^d \Lambda_t^\top B_t\Big) e_i (v_j v_j^\top), \\ &&& \qquad \forall\, i \in [k] \\ &&& \Lambda_t e_j \in C_t^*, \quad \forall\, (j,t) \in [p] \times [d] \\ &&& \mu \geq 0. \end{aligned}$$

Let $\overline{u}_r^{opt}$ denote the optimal value of the above program. It holds that $\overline{u}_r^{opt} \geq \overline{u}_{r+1}^{opt} \geq v^{opt}$ for each hierarchy level $r \in \mathbf{N}$.

The key results from the previous two sections are summarized in Theorem 3.

Theorem 3. *For each hierarchy level $r \in \mathbf{N}$, it holds that*

$$\ell_r^{opt} \leq v^{opt} \leq u_r^{opt} \leq \overline{u}_r^{opt}.$$

The practical value of the results from the previous two sections derives from the fact that one can obtain a feasible solution to the robust semidefinite program $\mathcal{P}$ by solving a finite-dimensional conic linear program, (16) or (19); and can bound the suboptimality incurred by this feasible solution by solving another finite-dimensional conic linear program (13).

VI. Numerical Study

We investigate the strengths and limitations of the proposed approach with a detailed numerical study.

A. Problem Description

As a concrete example, we consider the robust resistance network design problem studied in [6], [15]. A resistance network is an electrical network comprised of resistors and current sources. The network is modeled by a directed graph $G = ([n], E)$, where $[n]$ is the set of nodes and E is the set of directed edges. We let $(i, j) \in E$ if nodes i and j are connected by a resistor and $i < j$. Some nodes are assumed to be connected to ground. We let n_ϕ denote the number of nodes that are not connected to ground.

Given a fixed circuit topology, the objective is to design a set of resistances that minimize the maximal dissipation of the circuit (i.e., the energy consumed by the network), where the maximum is taken over a given set U of external currents. The set of external currents is given by

$$U = \{Q\xi \mid \xi \in \Xi\},$$

where the matrix $Q \in \mathbf{R}^{n_\phi \times k}$ is given, and Ξ is a given parameter uncertainty set satisfying the assumptions in Section II-A. The robust resistance network design problem can be cast as the following robust semidefinite program[7]

$$\begin{aligned} &\text{minimize} && \tau && (20)\\ &\text{subject to} && \tau \in \mathbf{R},\ g \in \mathbf{R}^{|E|} \\ &&& g \geq 0, \\ &&& \mathbf{1}^\top g \leq \omega, \\ &&& \begin{bmatrix} \tau & (Q\xi)^\top \\ Q\xi & A(g) \end{bmatrix} \in \mathbf{S}_+^{n_\phi+1}, \quad \forall\, \xi \in \Xi. \end{aligned}$$

Here, $g \in \mathbf{R}^{|E|}$ denotes the vector of conductances, $\mathbf{1} \in \mathbf{R}^{|E|}$ denotes the vector of all ones, $\omega \geq 0$ is a given budget parameter, and $A(g) := M\, \mathrm{diag}(g) M^\top$, where M is the $n_\phi \times |E|$ submatrix of the incidence matrix of the graph G. It is obtained by eliminating all rows associated with nodes that are connected to ground.

We consider the specific network described by the graph in Figure 1. Nodes 1 and 2 are assumed to be connected to ground. Thus, $n_\phi = 3$. We let $\omega = 9$, and set the matrix $Q \in \mathbf{R}^{3\times 6}$ equal to

$$Q = \begin{bmatrix} 0 & 4 & 1 & 1.2001 & 0 & 0 \\ 0 & 0.020 & 0.09 & 0 & 1.2001 & 0 \\ 0 & -0.050 & -0.2 & 0 & 0 & 1.2001 \end{bmatrix}.$$

In what follows, we apply the inner and outer hierarchies developed in this paper to the robust semidefinite program (20) for three different forms of the uncertainty set Ξ. We compare the performance of our technique with existing results from the literature [2], [6], [7], which were described in Section II-B.[8] All numerical analyses were carried out

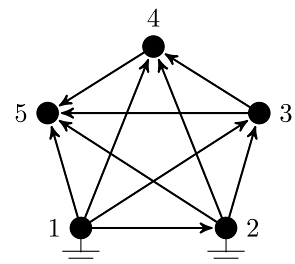

Fig. 1: The graph G describing the electrical network. Nodes 1 and 2 are connected to ground.

using *Gurobi Optimizer 6.5.0* and semidefinite programs were solved using *SDPT3* [16].

B. Unstructured Normed-Bounded Uncertainty

We consider first the case of unstructured norm-bounded uncertainty. More precisely, we let

$$\Xi \;=\; \{\xi \in \mathbf{R}^6 \mid \|\xi\|_2 \leq 2,\ \xi_1 = 1\}.$$

Theorem 2 indicates that uncertainty sets of this form enable the equivalent reformulation of the robust semidefinite program (20) as a semidefinite program. In particular, it is straightforward to show that an optimal solution to (20) can be obtained by solving the following semidefinite program implied by Theorem 2:

$$\begin{aligned} &\text{minimize} && \tau && (21)\\ &\text{subject to} && \tau \in \mathbf{R},\ g \in \mathbf{R}^{|E|} \\ &&& g \geq 0, \\ &&& \mathbf{1}^\top g \leq \omega, \\ &&& \begin{bmatrix} \tau I_k & Q^\top \\ Q & A(g) \end{bmatrix} \in \mathbf{S}_+^{n_\phi+k}. \end{aligned}$$

Here, I_k is the $k \times k$ identity matrix. The optimal value of (20) is equal to 2.37. And, as is shown in Table I, the sum of squares inner approximation method of [7] yields a feasible solution that achieves this optimal value. We also list in Table I the optimal values of the programs $\mathcal{P}_r^O$, $\mathcal{P}_r^I$, and $\overline{\mathcal{P}}_r^I$ for different levels r in the hierarchy. The corresponding finite-dimensional programs (13), (16), and (19) are second-order cone programs. Notice that the sequence of optimal values associated with the outer polyhedral hierarchy $\mathcal{P}_r^O$ nearly converge to the optimal value of the robust SDP (20) within the first few levels of the hierarchy.

C. Structured Normed-Bounded Uncertainty

Next, we consider the case of structured norm-bounded uncertainty. More precisely, we let

$$\Xi = \{\xi \in \mathbf{R}^6 \mid \|(\xi_2, \xi_3)\|_2 \leq 1,\ \|(\xi_4, \xi_5, \xi_6)\|_2 \leq 1,\ \xi_1 = 1\}.$$

Theorem 2 provides a conservative semidefinite program to calculate a feasible solution to (20) given uncertainty sets with structured norm-bounds. However, the resulting semidefinite program turns out to be infeasible for the specific

[7]We refer the reader to [15] for a detailed derivation of the semidefinite programming formulation of this problem.

[8]In applying the method from [7], feasible solutions were calculated with respect to the monomial bases $u_0(\xi) = (1, \xi)$ and $u(\xi) = 1$.

	Unstructured Normed-Bounded					Structured Normed-Bounded					Polytopic				
	r					r					r				
	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
$\mathcal{P}_r^O$	0	0	2.25	2.34	2.36	0	1.65	3.66	4.19	4.24	0	3.40	8.17	8.17	8.17
$\mathcal{P}_r^I$	∞	4.75	3.15	$*$	$*$	∞	6.35	5.26	$*$	$*$	∞	8.96	8.44	$*$	$*$
$\overline{\mathcal{P}}_r^I$	∞	6.72	4.94	4.56	4.55	∞	8.02	6.80	6.61	6.51	∞	8.96	8.44	8.34	8.26
Thm. 1			N/A					N/A					8.20		
Thm. 2			2.37					∞					N/A		
Thm. 1 [7]			2.37					4.27					8.22		

TABLE I: The table lists optimal values of the outer approximation, $\mathcal{P}_r^O$, and the inner approximations, $\mathcal{P}_r^I$ and $\overline{\mathcal{P}}_r^I$, for different levels r in the hierarchy. The corresponding finite-dimensional conic linear programs (13), (16), and (19) are second-order cone programs when the uncertainty set is norm-bounded, and linear programs when it is polytopic. The notation '*' indicates levels in the hierarchy for which the computation of the H-representation of the polyhedral cone I_r^n did not complete within three hours.

problem being studied, and therefore it provides no useful information. Alternatively, the approximation method of [7] yields a feasible solution having cost equal to 4.27. And, the lower bound of 4.24 implied by the outer polyhedral approximation $\mathcal{P}_r^O$ ($r = 4$) implies that this feasible solution is within a few percent of optimal.

D. Polytopic Uncertainty

Finally, we consider the case of polytopic uncertainty. More precisely, we let

$$\Xi \;=\; \{\xi \in \mathbf{R}^6 \mid \|\xi\|_\infty \leq 1,\ L\xi \geq 0,\ \xi_1 = 1\}.$$

Here, $L \in \mathbf{R}^{3\times 6}$ is a random matrix whose entries are sampled from the standard normal distribution. It is given by

$$L = \begin{bmatrix} 0.142 & 0.538 & 0.862 & -0.434 & 2.769 & 0.725 \\ 0.422 & 1.834 & 0.319 & 0.343 & -1.35 & -0.063 \\ 0.916 & -2.259 & -1.308 & 3.578 & 3.035 & 0.715 \end{bmatrix}.$$

In Table I, we list the optimal values of $\mathcal{P}_r^O$, $\mathcal{P}_r^I$, and $\overline{\mathcal{P}}_r^I$ for different levels r in the hierarchy. The corresponding finite-dimensional programs (13), (16), and (19) are linear programs. Notice that, for $r \geq 3$, $\overline{\mathcal{P}}_r^I$ yields a lower cost solution than $\mathcal{P}_2^I$. The last column of Table I implies that the the optimal value of the robust semidefinite program (20) must lie between 8.17 and 8.26. Indeed, by enumerating all vertices of the uncertainty set Ξ and applying Theorem 1, we are able to verify that the optimal value is equal to 8.20.

VII. Conclusion and Future Work

In this paper, we have investigated the problem of approximating solutions to intractable robust semidefinite programs. We proposed a hierarchy of inner and outer approximations for robust semidefinite programs, which are exact in the limit. The proposed approximation scheme becomes impractical for moderate levels in the hierarchy. This derives from the large number of constraints, which arise due to the polyhedral approximation of the positive semidefinite (PSD) cone. There are a variety of interesting directions for future work. One interesting direction would be to explore alternative polyhedral approximations of the PSD cone that rely on more intelligent (or adaptive) discretizations of the boundary of the cross polytope.

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