Networked Cournot Competition in Platform Markets: Access Control and Efficiency Loss

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***Abstract*—This paper studies network design and efficiency loss in online platforms using the model of networked Cournot competition. We consider two styles of platforms: open access platforms and discriminatory access platforms. In open access platforms, every firm can connect to every market, while discriminatory access platforms limit connections between firms and markets in order to improve upon social welfare. Our results provide tight bounds on the efficiency loss of both open access and discriminatory access platforms. In the case of open access platforms, we show that the efficiency loss at a Nash equilibrium is upper bounded by 3/2. In the case of discriminatory access platforms, we prove that under an assumption on the linearity of cost functions, a greedy algorithm for optimizing network connections can guarantee the efficiency loss at a Nash equilibrium is upper bounded by 4/3.**

I. INTRODUCTION

In recent years, online platforms have emerged as a dominant force. Platforms have changed the way entire industries are run, e.g., ride-sharing and online shopping. Unlike traditional firms, platforms themselves do not manufacture products or provide a service. Instead, they arrange matches between firms and consumers, and facilitate a safe and simple trading process, which provides value for all parties involved. Today, platforms such as Facebook, Uber, Amazon, Ebay, etc, make up a $3 trillion market in the US alone [1].

The design and operation of platforms is extremely diverse. Platforms like Amazon aim to *match* buyers to sellers in a clever way, taking into account sellers' prices and reviews and also similarly, buyers' buying preferences and locations, which determine shipping costs. Others use *pricing* to encourage the correct firm-market pairs to happen naturally, e.g., ad exchanges. In the extreme, some recent platforms have moved toward directly controlling the exact allocation of firms to markets, e.g., Uber chooses explicit matches and prices for drivers and riders [9, 11].

Broadly, there are two common schools of thought in platform design: (i) *Open access*, where the platform provides information on all potential matches, and allows firms and markets to make their own choices on their matching and the corresponding allocations [17, 18, 20], or (ii) *Discriminatory access*, where the platform restricts the set of markets each firm is allowed to enter in order to promote matches that are economically efficient [4, 12, 26]. Examples of open access platforms include eBay and Etsy, and examples of discriminatory access platforms include Amazon's Buy Box, the default seller of an item with respect to a particular buyer. We summarize these two approaches in Figure 1.

Open access and discriminatory access designs are contrasting approaches with differing benefits. Open access designs are simpler to maintain, completely transparent, and provide fairness across firms [27]. On the other hand, discriminatory access offers the platform additional control to optimize social welfare, at the expense of complexity, transparency, and fairness.

Thus, the question is, *how large an improvement in efficiency is possible by moving from open access to discriminatory access?* In other words, is the improvement in efficiency large enough to warrant the extra complexity, the loss of transparency, and the loss of fairness?

A. Contributions of this paper

In this paper we provide tight efficiency results for both open access and discriminatory access platform designs; thus quantifying the improvements in efficiency that discriminatory access designs can provide. Concretely, this paper builds on recent work [25], that studies platform design using the model of networked Cournot competition. In the context of this model, this paper makes two main contributions.

First, in Section III, we study the efficiency loss in open access platforms consisting of n firms. We provide a tight bound of $\frac{3}{2}(1-1/(3n+6))$ on the efficiency loss in open access platforms in Theorem 4, which improves upon the previously known $16/7$ efficiency loss bound [25]. Additionally, we provide a sharper efficiency loss bound in Proposition 6 that depends not only on the number of firms, but also the degree of asymmetry between firms' cost functions. In particular, this bound reveals that a reduction in the asymmetry between firms leads to a reduction in efficiency loss.

Second, in Section IV, we illustrate the efficiency improvement discriminatory access platforms provide over open access ones. Specifically, we consider a setting in which a discriminatory access platform solves an optimal network design problem to maximize the social welfare at the resulting Nash equilibrium. This amounts to a mathematical program with equilibrium constraints (MPEC) that is, in general, computationally intractable. Under the simplifying assumption that firms have linear cost functions, we construct a greedy algorithm that is guaranteed to yield an optimal network. Moreover, we provide a tight bound on the efficiency loss associated with said optimal network. Specifically, we show

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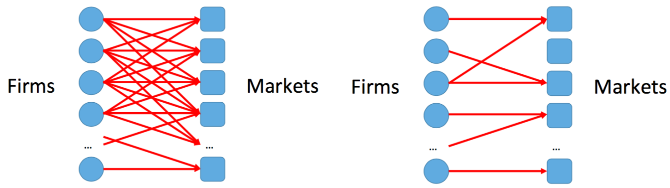

(a) Open access platforms. (b) Discriminatory access platforms.

Fig. 1: The above figures depict (a) open access platforms, where firms can participate in all markets, and (b) discriminatory access platforms, where the platform constrains the markets in which firms can participate. In both platforms, each firm can only access markets that it connects to via the red links, but can choose the exact quantity it allocates to each connected market strategically.

that discriminatory access platforms designed in this way yield an efficiency loss bounded by 4/3 in the worst case, as compared with the efficiency loss of 3/2 under open access.

B. Related work

Our work lies in the intersection of platform design and networked competition, contributing to both literatures.

a) Platform design: Recent growth of online platforms has led researchers to focus on identifying design features common to successful platforms. Work in this area has covered a variety of possible design factors, including pricing [28] and competition [6]. Recent empirical findings display significant price dispersion in online marketplaces [15], causing platforms to differentiate products in order to create distinct consumer markets [14]. In particular, these results highlight the need to study platforms in the context of *networked competition*.

b) Competition in networked settings: Models of networked competition aim to capture the effect of network constraints on the strategic interaction between firms. These models include networked Bertrand competition, e.g., [5, 10, 19], networked Cournot competition, e.g., [2, 7, 21], and various other non-cooperative bargaining games where agents can trade via bilateral contracts and a network determines the set of feasible trades, e.g., [3, 16, 24].

Our work fits squarely into the setting of networked Cournot competition. A large swath of literature on networked Cournot competitions, e.g., [2, 7, 21], focuses on characterizing and computing Nash equilibria. Recent streams of literature that closely relate to our work are (i) characterizing the efficiency loss of networked Cournot games [22, 25] and (ii) understanding the impacts of system operator governance on the resulting Nash equilibria [8]. This paper is the first to provide a tight bound on the efficiency loss of open access platforms, improving on the bounds in [25], and the first to provide an algorithm for network design with provable guarantees.

II. Model and Preliminaries

We describe competition in online platforms according to the *networked Cournot competition* model first introduced by [2] and [7], and later employed by [25] to describe competition in platforms. As a generalization of the classical model of Cournot competition, the networked Cournot model captures the setting in which firms compete to produce a homogeneous good in *multiple markets*, where each market is accessible by a subset of firms. We formally develop the model in the following subsections.

A. Network and Platform Models

The network specifying the connections between firms and markets is described according to a bipartite graph $(F, M, \mathcal{E})$. Here, we denote by $F := \{1, \dots, n\}$ the set of n firms, $M := \{1, \dots, m\}$ the set of m markets, and $\mathcal{E} \subseteq F \times M$ the set of directed edges connecting firms to markets. That is to say, $(i, j) \in \mathcal{E}$ if and only if firm i has access to market j.

In general, the efficiency of such marketplaces depends on the structure of the underlying graph, which restricts the set of markets to which each firm has access. A crucial role that the platform might therefore play in this setting is the selection of markets that are made available to each firm. In what follows, we examine two important classes of platform designs: *open access* platforms and *discriminatory access* platforms.

Open access platforms: An open access platform allows all firms to access all markets. This corresponds to the complete set of directed edges from firms to markets, which we denote by $\mathcal{K}_{F,M} = F \times M$. Examples of open access platforms include eBay and Etsy, where every customer is shown every retailer that sells the item he desires.

Discriminatory access platforms: In contrast to open access platforms, a discriminatory access platform can restrict the set of markets accessible by each firm. This corresponds to the platform's selection of an edge set $\mathcal{E} \subseteq F \times M$ that may prevent certain firms from accessing certain markets. The goal of this restriction is either to improve revenue or the social welfare of the system. An example of a discriminatory access platform is Amazon's Buy Box, where Amazon chooses a default seller based on a score combining pricing, availability, fulfillment, and customer service.

B. Producer Model

Under both the open and discriminatory access platform models, each firm i can specify the quantity it produces in each market. Accordingly, we let $q_{ij} \in \mathbb{R}_+$ denote the quantity produced by firm i in market j, and let $q_i := (q_{i1}, \dots, q_{im}) \in \mathbb{R}^m_+$ denote the supply profile from firm i. We require that $q_{ij} = 0$ for all $(i, j) \notin \mathcal{E}$, and define the set of feasible supply profiles from firm i as:

$$\mathcal{Q}_i(\mathcal{E}) := \left\{ x \in \mathbb{R}^m_+ \mid x_j = 0, \ \forall \ (i,j) \notin \mathcal{E} \right\}.$$

We denote the supply profile from all firms by $q := (q_1, \dots, q_n) \in \mathbb{R}^{mn}_+$. Accordingly, the set of feasible supply profiles from all firms is given by $\mathcal{Q}(\mathcal{E}) := \prod_{i=1}^n \mathcal{Q}_i(\mathcal{E})$.

The production cost of each firm $i \in F$ depends on its supply profile only through its aggregate production quantity, which is given by

$$s_i := \sum_{j=1}^m q_{ij}. \tag{1}$$

The production cost of firm i is defined by $C_i(s_i)$, where we assume that the cost function C_i is convex, differentiable on $(0, \infty)$ and satisfies $C_i(s_i) = 0$ for all $s_i \leq 0$.[1] Finally, we define $C := (C_1, \ldots, C_n)$ as the cost function profile.

C. Market Model

As is standard in Cournot models of competition, we model price formation according to an inverse demand function in each market. Following [7], we focus on affine inverse demand functions throughout this paper. Specifically, the price in each market $j \in M$ is determined according to

$$p_j(d_j) := \alpha_j - \beta_j d_j,$$

where d_j denotes the aggregate quantity being produced in market j. It is given by

$$d_j := \sum_{i=1}^{n} q_{ij}. \tag{2}$$

Here, $\alpha_j > 0$ measures consumers' maximum willingness to pay, and $\beta_j > 0$ measures the price elasticity of demand.

D. The Networked Cournot Game

We describe the equilibrium of the market specified above according to Nash. In particular, we consider profit maximizing firms, where the profit of a firm i, given the supply profiles of all other firms $q_{-i} = (q_1, .., q_{i-1}, q_{i+1}, .., q_n)$, is given by

$$\pi_i(q_i, q_{-i}) := \sum_{j=1}^{m} q_{ij} p_j(d_j) - C_i(s_i). \tag{3}$$

We denote by $\pi := (\pi_1, \ldots, \pi_n)$ the collection of payoff functions of all firms.

The triple $(F, \mathcal{Q}(\mathcal{E}), \pi)$ defines a normal-form game, which we refer to as the *networked Cournot game* associated with the edge set $\mathcal{E}$. Its Nash equilibrium is defined as follows.

Definition 1. A supply profile $q \in \mathcal{Q}(\mathcal{E})$ constitutes a *pure strategy Nash equilibrium* of the game $(F, \mathcal{Q}(\mathcal{E}), \pi)$ if for every firm $i \in F$, $\pi(q_i, q_{-i}) \geq \pi(\overline{q}_i, q_{-i})$, for all $\overline{q}_i \in \mathcal{Q}_i(\mathcal{E})$.

Under the assumptions of convex cost functions and affine inverse demand functions, [2] has shown that the networked Cournot game is an ordinal potential game, and the Nash equilibrium is the unique optimal solution to a convex program. Consequently, the networked Cournot game associated with any edge set $\mathcal{E} \subseteq F \times M$ admits a unique Nash equilibrium. We summarize the result in [2] in the following lemma.

Lemma 1. *([2]) The game $(F, \mathcal{Q}(\mathcal{E}), \pi)$ admits a unique Nash equilibrium $q^{NE}(\mathcal{E})$ that is the unique optimal solution to the following convex program:*

$$\underset{q \in \mathcal{Q}(\mathcal{E})}{\textit{maximize}} \quad \mathrm{SW}(q, C) - \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\beta_j q_{ij}^2}{2}. \tag{4}$$

[1]This family of cost functions represents a generalization of [25], which assumed that all firms have quadratic cost functions.

E. Social Welfare and the Price of Anarchy

In this paper, we measure the performance (or efficiency) of a platform according to *social welfare*. For platforms, the pursuit of social welfare benefits both buyers and sellers, and in the long run, promotes their expansion. For example, Amazon (in its Buy Box design) believes that welfare measures such as availability, fulfillment, and customer service ultimately lead to increased customer satisfaction, and thereby, promote its growth in the long run [13].

We adopt the standard measure of social welfare defined as sum of the aggregate consumer and producer surplus. The social welfare associated with a supply profile q and a cost function profile C is defined according to

$$\mathrm{SW}(q, C) := \sum_{j=1}^{m} \int_0^{d_j} p_j(z) dz - \sum_{i=1}^{n} C_i(s_i), \tag{5}$$

where s_i and d_j are defined in Eq. (1) and (2), respectively.

Further, we define the *efficient social welfare* associated with an edge set $\mathcal{E}$ and a cost function profile C as:

$$\mathrm{SW}^*(\mathcal{E}, C) := \sup_{q \in \mathcal{Q}(\mathcal{E})} \mathrm{SW}(q, C). \tag{6}$$

A supply profile $q \in \mathcal{Q}(\mathcal{E})$ is said to be *efficient* if it satisfies $\mathrm{SW}(q, C) = \mathrm{SW}^*(\mathcal{E}, C)$. It is straightforward to check that the above supremum can be attained, and that the set of efficient supply profiles is non-empty.

In general, the supply profile at the unique Nash equilibrium of the networked Cournot game will deviate from the efficient supply profile. We measure this loss of efficiency according the so called *price of anarchy* of the game [23].[2]

Definition 2. The *price of anarchy* associated with the edge set $\mathcal{E}$, the cost function profile C, and the corresponding networked Cournot game $(F, \mathcal{Q}(\mathcal{E}), \pi)$ is defined as

$$\rho(\mathcal{E}, C) := \frac{\mathrm{SW}^*(\mathcal{E}, C)}{\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}), C\right)}.$$

We set $\rho(\mathcal{E}, C) = 1$ if $\mathrm{SW}^*(\mathcal{E}, C)/\,\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}), C) = 0/0$.

III. Open Access Platforms

For our first set of results, we focus on providing tight bounds on the price of anarchy of the networked Cournot game in an open access platform, under a variety of assumptions on firms' cost functions. In particular, our tight price of anarchy bounds depend not only on the number of firms, but also on the degree of asymmetry between firms' cost functions. These results improve upon the bounds in [25] and generalize those in [22].

In what follows, we first establish a technical lemma, which reveals that the price of anarchy is maximized at a cost function profile consisting of cost functions that are linear over

[2]Implicit in our definition of the price of anarchy for the networked Cournot game is the fact that the networked Cournot game admits a unique Nash equilibrium. In general, for games with a possible multiplicity of Nash equilibria, the price of anarchy is defined as the ratio of the efficient social welfare over that of the Nash equilibrium with the *worst* social welfare.

the non-negative reals. Such result are useful in facilitating the derivation of tight bounds on the price of anarchy for networked Cournot games in an open access platform.

A. Identifying the Worst-case Cost Function Profile

The following lemma establishes piecewise linearity of the worst-case cost function profile. Its proof is deferred to the Appendix.

Lemma 2. *Given a cost function profile C, define the cost function profile $\overline{C} = (\overline{C}_1, \ldots, \overline{C}_n)$ according to*

$$\overline{C}_i(s_i) = \left(C_i' \left(\sum_{j=1}^{m} q_{ij}^{\mathrm{NE}}(\mathcal{K}_{F.M}) \right) s_i \right)^{+}$$

for $i = 1, \ldots, n$. It holds that $\rho(\mathcal{K}_{F,M}, C) \leq \rho(\mathcal{K}_{F,M}, \overline{C})$.

Lemma 2 reveals that, given any cost function profile C, it is always possible to construct another cost function profile $\overline{C}$ consisting of (piecewise) linear functions, which has a price of anarchy that is no smaller. Therefore, in constructing a price of anarchy bound that is guaranteed to hold for all cost functions belonging to the family specified in Section II-B, it suffices to consider cost functions that are linear on $(0, \infty)$.

B. Efficiency Loss in Open Access Platforms

The characterization of the worst-case cost function profile in Lemma 2 facilitates the derivation of tight upper bounds on the price of anarchy for networked Cournot games. In what follows, we examine the role played by *(a)symmetry* in the cost function profile in determining platform efficiency.

1) Symmetric Cost Functions: We begin by analyzing the setting in which firms have identical cost functions. Under this assumption, we establish a tight upper bound the price of anarchy in Proposition 3 that is *monotonically decreasing* in the number of firms, and converges to one as the number of firms grows large. This conforms with the intuition that increasing the number of (symmetric) suppliers will manifest in increased competition, and thereby reduce the extent to which any one producer might exert market power.

Proposition 3. *If $C_1 = C_2 = \cdots = C_n$, then the price of anarchy associated with the corresponding open access networked Cournot game $(F, \mathcal{Q}(\mathcal{K}_{F,M}), \pi)$ is bounded by*

$$\rho(\mathcal{K}_{F,M}, C) \leq 1 + \frac{1}{(n+1)^2 - 1}.$$

Moreover, the bound is tight. That is, for any choice of n, there exists a symmetric cost function profile with a corresponding price of anarchy equal to the upper bound.

The proof of Proposition 3 is provided in the Appendix.

2) Arbitrary Asymmetric Cost Functions: We now consider the more general setting in which firms have arbitrary asymmetric cost functions that satisfy the assumptions of Section II-B. In the following, we establish a tight upper bound on the price of anarchy that is *monotonically increasing* in the number of firms. Its proof is provided in the Appendix.

Theorem 4. *The price of anarchy associated with a cost function profile C and the corresponding open access networked Cournot game $(F, \mathcal{Q}(\mathcal{K}_{F,M}), \pi)$ is upper bounded by*

$$\rho(\mathcal{K}_{F,M}, C) \leq \frac{3}{2} \left(1 - \frac{1}{3n+6} \right).$$

The bound is tight if $\alpha_1 = \alpha_2 = \cdots = \alpha_m$.

The price of anarchy bound established in Theorem 4 is perhaps counterintuitive, in the sense that efficiency loss at a Nash equilibrium might be increasing in the number of firms. Consider the case in which the set of firms consist of one cheap firm and $n-1$ expensive firms. An increase in n manifests in both an increase in the aggregate supply and a reduction in supply from the cheap firm. If the resulting increase in consumer welfare is less than the increase in production cost, an increase in the number of firms n will manifest in a reduction in social welfare, and consequently, an increase in the price of anarchy.

Finally, we remark that taking the number of firms $n \to \infty$ yields a price of anarchy bound that is valid for any number of firms, and any number of markets. This recovers the $3/2$ price of anarchy bound first established by Johari and Tsitsiklis [22] for a single market. Moreover, it improves upon the previously known $16/7$ price of anarchy bound for open access networked Cournot games in [25]. We have the following corollary.

Corollary 5. *Open access platforms have a price of anarchy that is at most 3/2.*

Linear Cost Functions with Bounds on Asymmetry: The efficiency loss results in Proposition 3 and Theorem 4 appear contradictory. On the one hand, the price of anarchy bound is decreasing in n under the assumption of symmetric cost functions. However, when producers' cost functions are allowed to be asymmetric, the price of anarchy bound we derive is increasing in n. In what follows, we explore how the price of anarchy depends on the asymmetry between firms' cost functions. We restrict ourselves to cost functions that are linear on $(0, \infty)$ and whose slopes lie within $[c_{\min}, c_{\max}] \subseteq \mathbb{R}_+$.

$$\mathcal{L}(c_{\min}, c_{\max}) := \left\{ C_0 : \mathbb{R} \to \mathbb{R}_+ \;\middle|\; C_0(x) = (cx)^+ , \; c \in [c_{\min}, c_{\max}] \right\}.$$

We write $C \in \mathcal{L}^n(c_{\min}, c_{\max})$ if the cost function profile C satisfies $C_i \in \mathcal{L}(c_{\min}, c_{\max})$ for each firm $i \in F$. It will be convenient to define a non-dimensional parameter γ_j, which measures the degree of asymmetry between firms for each market $j \in M$. Specifically, for each market $j \in M$, define

$$\gamma_j := 1 - \frac{c_{\max} - c_{\min}}{\alpha_j - c_{\min}}.$$

It holds that $\gamma_j \in (-\infty, 1]$ when $c_{\min} < \alpha_j$ for each market $j \in M$. A large value of γ_j implies a small degree of asymmetry between firms' cost functions relative to consumers' maximum willingness to pay α_j in market j.

For this setting, we have the following proposition, which provides a tight price of anarchy bound for the networked Cournot game in an open access platforms, when firms have linear cost functions with bounded degrees of asymmetry. Its proof is deferred to the Appendix.

Proposition 6. *Let $C \in \mathcal{L}^n(c_{\min}, c_{\max})$, and assume that $c_{\min} < \max_{j\in M} \alpha_j$. The price of anarchy associated with the corresponding open access networked Cournot game $(F, \mathcal{Q}(\mathcal{K}_{F,M}), \pi)$ is upper bounded by*

$$\rho(\mathcal{K}_{F,M}, C) \leq \frac{\sum_{j=1}^m \dfrac{((\alpha_j - c_{\min})^+)^2}{\beta_j}}{\sum_{j=1}^m \left(\dfrac{2n+4}{3n+5} + \delta(\gamma_j, n)\right) \dfrac{((\alpha_j - c_{\min})^+)^2}{\beta_j}},$$

where the function $\delta(\gamma, n)$ is defined according to

$$\delta(\gamma, n) = \begin{cases} 0 & \text{if } \gamma < \frac{2n+3}{3n+5}, \\ \frac{(n-1)(3n+5)}{(n+1)^2}\left(\gamma - \frac{2n+3}{3n+5}\right)^2 & \text{otherwise} . \end{cases}$$

The bound is tight if $\alpha_1 = \alpha_2 = \cdots = \alpha_m$.

The price of anarchy bound specified in Proposition 6 depends on the degree of asymmetry between firms' cost functions only through the terms $\delta(\gamma_j, n)$ for $j = 1, \ldots, m$. In particular, as $\delta(\gamma, n)$ is non-decreasing in γ, a reduction in the degree of asymmetry between firms' cost functions can manifests in a reduction in the price of anarchy.

Finally, we remark that the price of anarchy bound in Proposition 6 may exhibit non-monotonic behavior in the number of firms n. More specifically, consider the case in which the maximum willingness to pay α_j is identical across all markets $j \in M$. When $\gamma \leq 17/22$, this price of anarchy bound is monotonically increasing in n. On the other hand, when $17/22 < \gamma < 1$, said price of anarchy bound decreases in n for $1 \leq n \leq \left\lfloor \frac{3\gamma-2}{1-\gamma} \right\rfloor$, and increases in n for $n \geq \left\lfloor \frac{3\gamma-2}{1-\gamma} \right\rfloor + 1$.

IV. Discriminatory Access Platforms

While many early platforms relied on an open access model, more recent platforms have begun to exercise control over the set of markets to which each firm has access. Here, the platform specifies the edge set of the bipartite graph that connect firms to markets with the goal of maximizing the social welfare at the unique Nash equilibrium of the resulting networked Cournot game. In doing this, the platform seeks to block access of each firm to markets where its participation might decrease market efficiency, i.e., decrease social welfare.

In what follows, we first show that the problem of choosing the optimal edge set that maximizes the social welfare at Nash equilibrium amounts to a mathematical program with equilibrium constraints (MPEC), and is, in general, computationally intractable. Under the simplifying assumption that each firm's cost function is linear on $(0, \infty)$, we present a greedy algorithm that is guaranteed to generate an optimal solution to the MPEC. Moreover, we present a tight price of anarchy bound for the resulting networked Cournot game. The bound reveals the reduction in efficiency loss achievable through discriminatory access platforms.

A. Network Design

The optimal network design problem amounts to the selection of an edge set $\mathcal{E}$, which maximizes the social welfare at the unique Nash equilibrium of the resulting networked Cournot game. Formally, Lemma 1 provides a characterization of the supply profile at the unique Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{E}), \pi)$ as the unique optimal solution to a convex program. Therefore, the *optimal network design problem* admits a formulation as the following MPEC:

$$\begin{aligned} \text{maximize} \quad & \mathrm{SW}(q, C) \\ \text{subject to} \quad & \mathcal{E} \subseteq F \times M \\ & q \in \underset{x \in \mathcal{Q}(\mathcal{E})}{\arg\max} \left\{ \mathrm{SW}(x, C) - \sum_{j=1}^m \sum_{i=1}^n \frac{\beta_j x_{ij}^2}{2} \right\} \end{aligned} \tag{7}$$

Here, the decision variables are the edge set $\mathcal{E}$ and the supply profile q. The challenge in solving problem (7) stems from the equilibrium constraint on q, and the presence of the combinatorial decision variable $\mathcal{E}$. In what follows, we show that, under the simplifying assumption of linear cost functions, the optimal network design problem (7) can be solved by a simple greedy algorithm.

B. A Greedy Algorithm for Linear Cost Functions

In this section we restrict ourselves to cost functions that are linear on $(0, \infty)$. More specifically, we assume that the cost function of each firm $i \in F$ satisfies $C_i(s_i) = (c_i s_i)^+$, where $c_i \geq 0$. Leveraging on this assumption, we propose a greedy algorithm for solving the optimal network design problem (7) in Algorithm 1. For each market $j \in M$, the greedy algorithm visits firms in ascending order of marginal cost, and provides each firm it visits access to market j if its inclusion in that market increases social welfare.

Algorithm 1 The Greedy Algorithm

Require: $c_1 \leq \cdots \leq c_n$.

Initialize edge set $\mathcal{E} \leftarrow \emptyset$.
for $j = 1$ to m **do**
  Initialize firm index $i \leftarrow 1$.
  Initialize edge set $\widetilde{\mathcal{E}} \leftarrow \mathcal{E}$.
  repeat
    Update edge set $\mathcal{E} \leftarrow \widetilde{\mathcal{E}}$.
    if $i \leq n$ **then**
      Set edge set $\widetilde{\mathcal{E}} \leftarrow \mathcal{E} \cup (i, j)$.
      Set firm index $i \leftarrow i + 1$.
    end if
  until $\mathrm{SW}(q^{\mathrm{NE}}(\widetilde{\mathcal{E}}), C) \leq \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}), C)$.
end for
return $\mathcal{E}$.

The following theorem establishes optimality of the greedy algorithm when firms' cost functions are linear over $(0, \infty)$. Its proof is provided in the Appendix.

Theorem 7. *Assume that each firm's cost function is linear over* $(0, \infty)$. *If* $\mathcal{E}^*$ *is the edge set generated by the greedy algorithm, then* $\left(\mathcal{E}^*, q^{\mathrm{NE}}(\mathcal{E}^*)\right)$ *is an optimal solution to* (7).

The implementation of Algorithm 1 yields an edge set $\mathcal{E}^*$, whose corresponding Nash equilibrium has a social welfare that is no smaller than that of the open access platform. In the following theorem, we quantify such an improvement in social welfare via a tight bound on the price of anarchy in discriminatory access networked Cournot games. Its proof is deferred to the Appendix.

Theorem 8. *Let* $C \in \mathcal{L}^n(c_{\min}, c_{\max})$*, and assume that* $c_{\min} < \max_{j \in M} \alpha_j$. *If* $\mathcal{E}^*$ *is the edge set generated by the greedy algorithm, then the efficient social welfare associated with the edge set* $\mathcal{E}^*$ *satisfies*

$$\mathrm{SW}^*(\mathcal{E}^*, C) = \mathrm{SW}^*(\mathcal{K}_{F,M}, C).$$

Moreover, the price of anarchy associated with the discriminatory access networked Cournot game $(F, \mathcal{Q}(\mathcal{E}^*), \pi)$ *is upper bounded by*

$$\rho(\mathcal{E}^*, C) \leq \frac{\sum_{j=1}^{m} \frac{\left((\alpha_j - c_{\min})^+\right)^2}{\beta_j}}{\sum_{j=1}^{m} \max\limits_{k \in \{1,..,n\}} \left\{\frac{2k+4}{3k+5} + \delta(\gamma_j, k)\right\} \frac{((\alpha_j - c_{\min})^+)^2}{\beta_j}}.$$

The above bound is tight if $\alpha_1 = \alpha_2 = \cdots = \alpha_m$.

Theorem 8 reveals the advantage discriminatory access platforms have over open access ones in reducing the efficiency loss at Nash equilibrium. Namely, when the edge set is chosen to be an optimal solution of the network design problem (7), the discriminatory access platform is guaranteed to have a tight bound on the price of anarchy that is no larger than that of the open access platform. Moreover, this price of anarchy bound is guaranteed to be non-increasing in the number of firms n.

Additionally, as the degree of asymmetry between firms' cost functions decreases, the price of anarchy bound for discriminatory access platforms gets close to that of open access platforms. More specifically, we consider the choice of $k^* \in \{1, .., n\}$ that maximizes $\frac{2k+4}{3k+5} + \delta(\gamma, k)$, as a function of γ. If $\gamma \leq 17/22$, the above maximum is attained by $k^* = 1$; whereas if $17/22 < \gamma < 1$, k^* satisfies

$$k^* \in \left\{\min\left\{\left\lfloor\frac{3\gamma - 2}{1 - \gamma}\right\rfloor, n\right\},\ \min\left\{\left\lfloor\frac{3\gamma - 2}{1 - \gamma}\right\rfloor + 1, n\right\}\right\}.$$

Clearly, as γ increases, k^* increases. In particular, for a fixed number of firms n, if $\gamma_j \geq 1 - 1/(n+3)$ for each market $j \in M$, the price of anarchy bounds for open access and discriminatory access platforms become identical.

Finally, under the assumption of a linear cost function profile, a straightforward calculation involving the price of anarchy bound in Theorem 8 yields a price of anarchy bound of $4/3$ for (optimized) discriminatory access platforms with any number of firms and markets. It represents an improvement upon the $3/2$ price of anarchy bound for open access platforms established in Corollary 5. The result is formally stated as follows.

Corollary 9. *Assume that each firm's cost function is linear over* $(0, \infty)$. *Discriminatory access platforms have a price of anarchy of at most* $4/3$.

V. Concluding Remarks

This paper examines the design and efficiency loss of two styles of widely used platforms: open access platforms and discriminatory access platforms. Open access platforms provide transparency, while discriminatory access platforms provide additional control that might be leveraged on to improve market efficiency. For open access platforms, we establish a tight upper bound on the price of anarchy that is decreasing in the number of firms, when costs are symmetric. On the other hand, when costs are arbitrarily asymmetric, we derive a tight upper bound on the price of anarchy that is increasing in the number of firms, and show that open access platforms have a price of anarchy of at most 3/2.

Our second set of results contrast this bound with the case of discriminatory access platforms. We formulate the optimal network design problem for discriminatory access platforms as a mathematical program with equilibrium constraints (MPEC), which is, in general, computationally intractable. Under the simplifying assumption that the firms' costs are linear, we propose and prove the optimality of a greedy algorithm, recovering the optimal network design for discriminatory access platforms in networked Cournot games. In this setting, we show that the price of anarchy bound shrinks to 4/3, thereby improving upon the worst-case efficiency loss of open access platforms.

Our work builds on a growing literature studying networked Cournot competition, including [2, 7, 8, 21, 25]. While this literature is maturing, there are still a wide variety of important open questions that remain. For example, the formulation of the optimal network design as an MPEC highlights that they are, in general, difficult to solve. The problem of constructing approximation algorithms with provable bounds on performance arises as an interesting direction for future work. Additionally, all of our results rely on a worst-case analysis of efficiency loss in networked Cournot games. The question as to how large the efficiency loss might be in the average case remains an interesting open question.

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APPENDIX

This appendix provides proofs of all results in this paper.

A. Proof of Lemma 2

Let $q^{\mathrm{NE}}(\mathcal{K}_{F,M}) \in \mathcal{Q}(\mathcal{K}_{F,M})$ be the unique Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{K}_{F,M}), \pi)$ associated with an arbitrary cost function profile C. Throughout the proof, we always consider a networked Cournot game associated with the edge set $\mathcal{K}_{F,M}$. Thus, for notational simplicity we use q^{NE} instead of $q^{\mathrm{NE}}(\mathcal{K}_{F,M})$ for the remainder of the proof. For each $i \in F$, we define the scalar λ_i according to

$$\lambda_i := C_i'\left(\sum_{j=1}^{m} q_{ij}^{\mathrm{NE}}\right). \tag{8}$$

Here, λ_i is the marginal cost of firm i at the unique Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{K}_{F,M}), \pi)$. We define a (piecewise) linear cost function profile $\overline{C} = (\overline{C}_1, \ldots, \overline{C}_n)$ according to

$$\overline{C}_i(s_i) := \max\{\lambda_i s_i,\, 0\}, \quad i = 1, \ldots, n.$$

It is clear that this cost function profile satisfies the assumption we have on cost functions in Section II-B. The stationarity conditions of firms' profit maximization problem under the cost function profiles C and $\overline{C}$ are identical at q^{NE}. Thus, q^{NE} is the unique Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{K}_{F,M}), \overline{\pi})$ associated with the cost function profile $\overline{C}$. Our objective is to show that $\rho(\mathcal{K}_{F,M}, C) \le \rho(\mathcal{K}_{F,M}, \overline{C})$, i.e.,

$$\frac{\mathrm{SW}(q^{\mathrm{NE}}, C)}{\mathrm{SW}^*(\mathcal{K}_{F,M}, C)} \ge \frac{\mathrm{SW}(q^{\mathrm{NE}}, \overline{C})}{\mathrm{SW}^*(\mathcal{K}_{F,M}, \overline{C})}.$$

In showing this, we first define the scalar μ_i for each firm $i \in F$ according to

$$\mu_i := C_i'\left(\sum_{j=1}^{m} q_{ij}^{\mathrm{NE}}\right) \cdot \left(\sum_{j=1}^{m} q_{ij}^{\mathrm{NE}}\right) - C_i\left(\sum_{j=1}^{m} q_{ij}^{\mathrm{NE}}\right).$$

For each firm i, μ_i equals the absolute difference in his production cost at Nash equilibrium associated with the cost function profiles C and $\overline{C}$.

With the definition of μ_i in hand, we define an "intermediate" cost function profile $\widetilde{C} = (\widetilde{C}_1, \ldots, \widetilde{C}_n)$ as follows:

$$\widetilde{C}_i(s_i) := \max\{\lambda_i s_i - \mu_i,\, 0\}, \quad i = 1, \ldots, n.$$

The cost function profile $\widetilde{C}$ makes the connection between the cost function profiles C and $\overline{C}$. On one hand, for each firm $i \in F$, the cost function $\widetilde{C}_i$ can be regarded as a linearization of the cost function C_i around the Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{K}_{F,M}), \pi)$. On the other hand, it can be understood as a translation of the cost function $\overline{C}_i$ downwards along the y-axis of length μ_i, while keeping the resulting cost function nonnegative for all real numbers. It is straightforward to verify that q^{NE} is the unique Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{K}_{F,M}), \widetilde{\pi})$ associated with the cost function profile $\widetilde{C}$.

Since firms' production costs at q^{NE} are equal for the cost function profiles C and $\widetilde{C}$, $SW(q^{\mathrm{NE}}, C) = SW(q^{\mathrm{NE}}, \widetilde{C})$. On the other hand, since $C_i(s_i) \ge \widetilde{C}_i(s_i)$ for all $s_i \in \mathbb{R}_+$, $\mathrm{SW}^*(\mathcal{K}_{F,M}, C) \le \mathrm{SW}^*(\mathcal{K}_{F,M}, \widetilde{C})$. Consequently, we have

$$\frac{\mathrm{SW}(q^{\mathrm{NE}}, C)}{\mathrm{SW}^*(\mathcal{K}_{F,M}, C)} \ge \frac{\mathrm{SW}(q^{\mathrm{NE}}, \widetilde{C})}{\mathrm{SW}^*(\mathcal{K}_{F,M}, \widetilde{C})}. \tag{9}$$

Additionally, the previous translation from $\overline{C}$ to $\widetilde{C}$ implies that $SW(q^{\text{NE}},\widetilde{C})$ and $SW(q^{\text{NE}},\overline{C})$ are related according to:

$$\text{SW}(q^{\text{NE}},\overline{C}) = \text{SW}(q^{\text{NE}},\widetilde{C}) - \sum_{i=1}^{n}\mu_i \geq 0. \tag{10}$$

We claim that the following inequality holds for the efficient social welfare $\text{SW}^*(\mathcal{K}_{F,M},\overline{C})$ associated with the cost function profile $\overline{C}$:

$$\text{SW}^*(\mathcal{K}_{F,M},\overline{C}) \geq \text{SW}^*(\mathcal{K}_{F,M},\widetilde{C}) - \sum_{i=1}^{n}\mu_i.$$

To see this, let q^* be a social welfare maximizer under the cost function profile $\widetilde{C}$. For each $i \in F$, we have

$$\overline{C}_i\left(\sum_{j=1}^{m} q^*_{ij}\right) \leq \widetilde{C}_i\left(\sum_{j=1}^{m} q^*_{ij}\right) + \mu_i.$$

It follows that

$$\begin{aligned}\text{SW}^*(\mathcal{K}_{F,M},\overline{C}) &\geq \text{SW}(q^*,\overline{C}) \geq \text{SW}(q^*,\widetilde{C}) - \sum_{i=1}^{n}\mu_i \\ &= \text{SW}^*(\mathcal{K}_{F,M},\widetilde{C}) - \sum_{i=1}^{n}\mu_i.\end{aligned}$$

The above inequality, in combination with inequalities (9) and (10), shows that

$$\begin{aligned}&\frac{\text{SW}(q^{\text{NE}},C)}{\text{SW}^*(\mathcal{K}_{F,M},C)} \geq \frac{\text{SW}(q^{\text{NE}},\widetilde{C})}{\text{SW}^*(\mathcal{K}_{F,M},\widetilde{C})} \\ &\geq \frac{\text{SW}(q^{\text{NE}},\widetilde{C}) - \sum_{i=1}^{n}\mu_i}{\text{SW}^*(\mathcal{K}_{F,M},\widetilde{C}) - \sum_{i=1}^{n}\mu_i} \geq \frac{\text{SW}(q^{\text{NE}},\overline{C})}{\text{SW}^*(\mathcal{K}_{F,M},\overline{C})},\end{aligned}$$

as needed to be shown.

B. *Proof of Proposition 3*

Using Lemma 2, one can show that the worst symmetric cost function profile that maximizes $\rho(\mathcal{K}_{F,M},C)$ consists of n identical cost functions that are linear on $(0,\infty)$. Thus, to upper bound the PoA of the networked Cournot game, it suffices to consider a cost function profile $\overline{C}$ that satisfies

$$\overline{C}_i(x) = cx \quad \text{for } x > 0, \quad \text{for all } i \in F, \tag{11}$$

for a finite positive constant $c > 0$ that is independent of i.

Given the assumption on the (piecewise) linearity of cost, it is straightforward to show that the unique Nash equilibrium and an efficient supply profile of the corresponding networked Cournot game are respectively given by

$$q^{\text{NE}}_{ij} = \frac{(\alpha_j - c)^+}{\beta_j(n+1)}, \quad q^*_{ij} = \frac{(\alpha_j - c)^+}{\beta_j n}, \quad \text{for } i \in F,\ j \in M.$$

It follows that the social welfare at the unique Nash equilibrium q^{NE} of this Cournot game is given by

$$SW^*(q^{\text{NE}},\overline{C}) = \sum_{j=1}^{m}\frac{\left((\alpha_j - c)^+\right)^2}{2\beta_j}\left(1 - \frac{1}{(n+1)^2}\right).$$

And the efficient social welfare is given by

$$SW^*(\mathcal{K}_{F,M},\overline{C}) = \sum_{j=1}^{m}\frac{\left((\alpha_j - c)^+\right)^2}{2\beta_j}.$$

Hence, the price of anarchy associated with the cost function profile $\overline{C}$ is given by

$$\rho(\mathcal{K}_{F,M},\overline{C}) = \begin{cases} 1 + \frac{1}{(n+1)^2-1} & \text{if } \max_{j\in M}\alpha_j > c, \\ 1 & \text{otherwise}\end{cases}.$$

Choosing $c < \max_{j\in M}\alpha_j$ gives the worst-case cost function profile that maximizes the price of anarchy over symmetric cost function profiles. This completes the proof.

C. *Proof of Theorem 4*

As is implied by Lemma 2, in order to construct a bound on the price of anarchy, it suffices to restrict to cost functions that are linear on $(0,\infty)$. We thus assume that each firm's cost function satisfies $\overline{C}_i(s_i) = (c_i s_i)^+$, for $i = 1,\ldots,n$, where we assume without loss of generality that $c_1 \leq \cdots \leq c_n$.

Note that, if $n = 1$, one can show that for any cost function profile C, there exists a linear cost function profile $\overline{C}$, such that $\rho(\mathcal{K}_{F,M},C) \leq \rho(\mathcal{K}_{F,M},\overline{C}) = \frac{4}{3}$. Thus, for the remainder of the proof we restrict ourselves to $n \geq 2$.

We first consider the simple setting where the number of markets $m = 1$. Without loss of generality, we assume that $c_1 < \alpha_1$. Since cost functions are linear, there exists an efficient supply profile q^* that assigns all production to firm 1, i.e., $q^*_{i1} = 0 \quad$ for $i = 2,\ldots,n$. The supply from firm 1 and the corresponding efficient social welfare obtained is:

$$q^*_{11} = \frac{\alpha_1 - c_1}{\beta_1}, \quad \text{and} \quad SW^*(\mathcal{K}_{F,\{1\}},\overline{C}) = \frac{(\alpha_1 - c_1)^2}{2\beta_1}.$$

Fixing α_1, β_1, c_1, we can optimize over $c_2,\ldots,c_n$ in order to minimize the social welfare at the unique Nash equilibrium of the Cournot game. This problem can be formulated as a symmetric convex program over the production quantities of the remaining firms at Nash equilibrium. One can show that the corresponding optimal values of $c_2,\ldots,c_n$ is:

$$c^*_i = \alpha_1 - \frac{2n+3}{3n+5}(\alpha_1 - c_1), \quad \text{for } i = 2,\ldots,n.$$

And the production quantity of each producer is given by

$$q_{i1} = \begin{cases} \frac{\alpha_1 - c_1}{\beta_1}\cdot\frac{n+3}{3n+5} & \text{if } i = 1. \\ \frac{\alpha_1 - c_1}{\beta_1}\cdot\frac{1}{3n+5}, & \text{if } i = 2,\ldots,n.\end{cases}$$

Define the cost function profile $\overline{C}^* = (\overline{C}^*_1,\ldots,\overline{C}^*_n)$ according to $\overline{C}^*_1(s_1) = (c_1 s_1)^+$ and $\overline{C}^*_i(s_i) = (c^*_i s_i)^+$ for $i = 2,\ldots,n$. Thus, for the fixed parameters α_1,β_1,c_1, the minimum social welfare at Nash equilibrium is given by

$$\text{SW}(q^{\text{NE}}(\mathcal{K}_{F,\{1\}}),\overline{C}^*) = \frac{(n+2)(\alpha_1 - c_1)^2}{(3n+5)\beta_1}.$$

It follows that for any linear cost function profile $\overline{C}$, we have

$$\rho(\mathcal{K}_{F,\{1\}},\overline{C}) \leq \rho(\mathcal{K}_{F,\{1\}},\overline{C}^*) = \frac{3}{2}\left(1 - \frac{1}{3n+6}\right).$$

We now consider the slightly more complicated setting in which $m > 1$. The efficient social welfare associated with a linear cost function profile $\overline{C}$ is given by

$$\mathrm{SW}^*(\mathcal{K}_{F,M}, \overline{C}) = \sum_{j=1}^{m} \frac{\left((\alpha_j - c_1)^+\right)^2}{2\beta_j}.$$

Given the linearity of firms' cost functions over $(0, \infty)$, the networked Cournot game decouples across markets. We have

$$\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{K}_{F,M}), \overline{C}) = \sum_{j=1}^{m} \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{K}_{F,\{j\}}), \overline{C})$$

$$\geq \sum_{j=1}^{m} \frac{\left((\alpha_j - c_1)^+\right)^2}{3\beta_j \left(1 - \frac{1}{3n+6}\right)} = \frac{\mathrm{SW}^*(\mathcal{K}_{F,M}, \overline{C})}{\frac{3}{2}\left(1 - \frac{1}{3n+6}\right)}.$$

It follows that for any cost function profile C, there exists a linear cost function profile $\overline{C}$, such that

$$\rho(\mathcal{K}_{F,M}, C) \leq \rho(\mathcal{K}_{F,M}, \overline{C}) \leq \frac{3}{2}\left(1 - \frac{1}{3n+6}\right).$$

D. Proof of Proposition 6

Due to space constraints, we only provide a proof of the price of anarchy bound, since the proof of tightness is more straightforward. The proof proceeds in two steps. First, we provide a price of anarchy bound for the simple setting in which the number of markets $m = 1$. Second, given the assumption on the linearity of the cost functions, we generalize this price of anarchy bound to the case in which $m > 1$. In particular, the key intuition of the proof is that, under a linear cost function profile, the networked Cournot game decouples into m Cournot games in each market.

Step 1: We provide a price of anarchy bound for the case in which the number of markets $m = 1$. Namely, using techniques similar to the proof of Theorem 4, one can establish the following price of anarchy bound for $C \in \mathcal{L}^n(c_{\min}, c_{\max})$

$$\rho(\mathcal{K}_{F,\{j\}}, C) \leq \frac{1}{\frac{2n+4}{3n+5} + \delta(\gamma_j, n)}, \quad j = 1, \ldots, m.$$

Step 2: Given the linearity of firms' cost functions on $(0, \infty)$, we generalize the price of anarchy bound developed in Step 1 to the case in which the number of markets $m > 1$. We begin by assuming that each firm i's cost function is given by

$$C_i(s_i) = (c_i s_i)^+, \quad \text{where } c_i \in [c_{\min}, c_{\max}].$$

Without loss of generality, we assume that $c_1 \leq \cdots \leq c_n$. Thus, the efficient social welfare associated with the edge set $\mathcal{K}_{F,M}$ and the cost function profile C is given by

$$\mathrm{SW}^*(\mathcal{K}_{F,M}, C) = \sum_{j=1}^{m} \frac{\left((\alpha_j - c_1)^+\right)^2}{2\beta_j}.$$

As before, since costs are linear, the networked Cournot game decouples, and the social welfare at the unique Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{K}_{F,M}), \pi)$ satisfies

$$\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{K}_{F,M}), C) = \sum_{j=1}^{m} \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{K}_{F,\{j\}}), C).$$

The right-hand-side (RHS) of the above equation satisfies:

$$\sum_{j=1}^{m} \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{K}_{F,\{j\}}), C) = \sum_{j=1}^{m} \frac{\mathrm{SW}^*(\mathcal{K}_{F,\{j\}}, C)}{\rho(\mathcal{K}_{F,\{j\}}, C)} \quad (12)$$

$$\geq \sum_{j=1}^{m} \frac{\left((\alpha_j - c_1)^+\right)^2}{2\beta_j} \left(\frac{2n+4}{3n+5} + \delta(\gamma_j, n)\right). \quad (13)$$

Here, (12) follows from the definition of the price of anarchy, and inequality (13) follows from the price of anarchy bound in Step 1. Inequality (13) provides the following lower bound on the reciprocal of the price of anarchy $\rho(\mathcal{K}_{F,M}, C)$:

$$\frac{1}{\rho(\mathcal{K}_{F,M}, C)} \geq \frac{\sum_{j=1}^{m} \left(\frac{2n+4}{3n+5} + \delta(\gamma_j, n)\right) \frac{\left((\alpha_j - c_1)^+\right)^2}{\beta_j}}{\sum_{j=1}^{m} \frac{\left((\alpha_j - c_1)^+\right)^2}{\beta_j}}.$$

By taking the derivative of the RHS of the above inequality with respect to c_1, we show that the RHS of the above inequality is monotonically non-decreasing in c_1 for $c_1 \in [c_{\min}, c_{\max}]$. Hence, choosing $c_1 = c_{\min}$ minimizes the RHS of the above inequality. This completes the proof.

E. Proof of Theorem 7

Without loss of generality, we assume that $c_1 \leq \cdots \leq c_n$. We only provide a proof for the case in which the number of market $m = 1$. We emphasize, however, that given the assumption on the linearity of firms' cost functions over $(0, \infty)$, the networked Cournot game decouples across markets. As a result, our proof readily generalizes to the case in which the number of markets $m > 1$.

We denote by $F_1(\mathcal{E})$ the set of firms that have access to market 1, when the edge set is given by $\mathcal{E} \subseteq F \times \{1\}$. More specifically, the set $F_1(\mathcal{E})$ is defined according to

$$F_1(\mathcal{E}) = \{i \in F \mid (i, 1) \in \mathcal{E}\}.$$

We first introduce the concept of a contiguous set of firms, which plays a central role in the remainder of the proof. More specifically, we say that the set $F_1(\mathcal{E})$ is *contiguous* if

$$F_1(\mathcal{E}) = \{1, \ldots, |F_1(\mathcal{E})|\}, \quad \text{or} \quad F_1(\mathcal{E}) = \emptyset.$$

Here, $|F_1(\mathcal{E})|$ denotes the cardinality of the set $F_1(\mathcal{E})$. Clearly, for the edge set $\mathcal{E}^*$ generated by the greedy algorithm, the set $F_1(\mathcal{E}^*)$ is contiguous.

The rest of the proof consists of two parts. In Part 1, we show that if the set $F_1(\mathcal{E})$ is contiguous, then the social welfare at the Nash equilibrium associated with edge set $\mathcal{E}$ is guaranteed to be no larger than that of the edge set $\mathcal{E}^*$. In Part 2, we consider the case in which the set $F_1(\mathcal{E})$ is not contiguous. We show that there exists an edge set $\widetilde{\mathcal{E}}$ that yields a contiguous set $F_1(\widetilde{\mathcal{E}})$, and has social welfare at Nash equilibrium that is no smaller than that of the edge set $\mathcal{E}$.

Part 1: In this part, we assume that the set $F_1(\mathcal{E})$ is contiguous, and show that $\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}), C) \leq \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}^*), C)$. We first define a sequence of edge sets according to

$$\mathcal{E}_k = \bigcup_{i=1}^{k} \{(i, 1)\}, \quad k = 0, \ldots, n. \quad (14)$$

Here, we emphasize that $\mathcal{E}_0 = \emptyset$. Let $k^* = |\mathcal{E}^*|$. To show that $\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}), C) \leq \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}^*), C)$ if $F_1(\mathcal{E})$ is contiguous, it suffices to show that the sequence $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right)$ is strictly increasing in k over $k = 0, \ldots, k^*$, and monotonically non-increasing in k over $k = k^*, \ldots, n$.

In order to show this, we assume without loss of generality that $c_i \leq \alpha_1$ for all i. If this is not the case, one can work with an alternative cost function profile $\widetilde{C} = (\widetilde{C}_1, \ldots, \widetilde{C}_n)$ that is defined according to $\widetilde{C}_i(s_i) = (\min\{c_i, \alpha_1\} s_i)^+$ for $i = 1, \ldots, n$. Clearly, $\min\{c_i, \alpha_1\} \leq \alpha_1$ for all i, and $\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k), C) = \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k), \widetilde{C})$ for all $k = 0, \ldots, n$.

The proof of this monotonicity result relies critically on the following lemma.

Lemma 10. *Assume that each firm i's cost function is of the form $C_i(s_i) = (c_i s_i)^+$, where $c_1 \leq \cdots \leq c_n$. Let the number of markets $m = 1$, and the edge set $\mathcal{E}_k$ be defined according to Eq.* (14). *For each $k = 1, \ldots, n$, we have that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right) > \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}),\, C\right)$ if and only if*

$$\alpha_1 - c_k > \frac{1}{k}\left(1 + \frac{1}{k - \frac{1}{2(k+1)}}\right)\left(\sum_{i=1}^{k-1}(\alpha_1 - c_i)\right). \quad (15)$$

The proof of Lemma 10 is deferred to Appendix G. According to the description of the greedy algorithm, we have that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right)$ is strictly increasing in k for $0 \leq k \leq k^*$. Additionally, we have that

$$\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k^*+1}),\, C\right) \leq \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k^*}),\, C\right).$$

It follows from Lemma 10 that for $k = k^* + 1$, the following inequality is satisfied:

$$\alpha_1 - c_k \leq \frac{1}{k}\left(1 + \frac{1}{k - \frac{1}{2(k+1)}}\right)\left(\sum_{i=1}^{k-1}(\alpha_1 - c_i)\right). \quad (16)$$

To complete Part 1 of the proof, we have the following lemma.

Lemma 11. *Let $k^* \in \{0, \ldots, n\}$, and assume that $c_1 \leq \cdots \leq c_n \leq \alpha_1$. If inequality* (16) *is satisfied for $k = k^* + 1$, then it is satisfied for $k = k^* + 1, \ldots, n$.*

The proof of Lemma 11 is deferred to Appendix H. A combination of Lemma 10 and 11 reveals that

$$\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right) \leq \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}),\, C\right)$$

for $k = k^* + 1, \ldots, n$. This completes Part 1 of the proof.

Part 2: In this part, we assume that the set $F_1(\mathcal{E})$ is not contiguous. We show that there exists an edge set $\widetilde{\mathcal{E}} \subseteq F \times \{1\}$, such that the set $F_1(\widetilde{\mathcal{E}})$ is contiguous, and $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}),\, C\right) \leq \mathrm{SW}\left(q^{\mathrm{NE}}(\widetilde{\mathcal{E}}),\, C\right)$.

Our proof of the above claim is constructive. Given an edge set $\mathcal{E} \subseteq F \times \{1\}$, we construct a sequence of $(n^2 + 1)$ edge sets according to the following procedure

1) Set $k = 0$, and $\mathcal{E}_k = \mathcal{E}$.
2) If the set $F_1(\mathcal{E}_k)$ is contiguous, then set $\mathcal{E}_{k+1} = \mathcal{E}_k$, and go to Step 5. If not, go to Step 3.
3) Define the edge set $\widetilde{\mathcal{E}}_k$ according to
$$\widetilde{\mathcal{E}}_k = \mathcal{E}_k \setminus \left\{\left(\max F_1(\mathcal{E}_k), 1\right)\right\}. \quad (17)$$
4) If $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right) \leq \mathrm{SW}\left(q^{\mathrm{NE}}(\widetilde{\mathcal{E}}_k),\, C\right)$, then set $\mathcal{E}_{k+1} = \widetilde{\mathcal{E}}_k$. If not, set
$$\mathcal{E}_{k+1} = \widetilde{\mathcal{E}}_k \cup \left\{\left(\min\left(F \setminus F_1(\mathcal{E}_k)\right), 1\right)\right\}. \quad (18)$$
5) If $k < n^2$, update $k = k+1$, and go to Step 2. If $k \geq n^2$, terminate the procedure.

We will show that both of the following claims are true

(i) The set $F_1(\mathcal{E}_{n^2})$ is contiguous.
(ii) For $k = 0, \ldots, n^2 - 1$, we have
$$\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right) \leq \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k+1}),\, C\right). \quad (19)$$

We emphasize that if the second claim is true, then we readily have that $\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}), C) \leq \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_{n^2}), C)$. In what follows, we show that Claim (i) and (ii) are true in Part 2.1 and 2.2 of the proof, respectively.

Part 2.1: Proof of Claim (i). We show that Claim (i) is true according to a "potential function" argument. Namely, we define a potential function on the edge set $\mathcal{E}_k \subseteq F \times \{1\}$ as follows

$$\Phi(\mathcal{E}_k) = |F_1(\mathcal{E}_k)|\left(\max F_1(\mathcal{E}_k) - |F_1(\mathcal{E}_k)|\right)$$

for $k = 0, \ldots, n^2$. For all $\mathcal{E}_k \subseteq F \times \{1\}$, we have that $\Phi(\mathcal{E}_k) \geq 0$. It is straightforward to show $\Phi(\mathcal{E}_k) = 0$ if and only if the set $F_1(\mathcal{E}_k)$ is contiguous. Additionally, if $\Phi(\mathcal{E}_k) > 0$, then either either $\mathcal{E}_{k+1} = \widetilde{\mathcal{E}}_k$, or $\mathcal{E}_{k+1}$ is specified according to Eq. (18). If $\mathcal{E}_{k+1} = \widetilde{\mathcal{E}}_k$, we have

$$\begin{aligned}
\Phi(\mathcal{E}_{k+1}) &= |F_1(\widetilde{\mathcal{E}}_k)|\left(\max F_1(\widetilde{\mathcal{E}}_k) - |F_1(\widetilde{\mathcal{E}}_k)|\right) && (20)\\
&= (|F_1(\mathcal{E}_k)| - 1)\left(\max F_1(\widetilde{\mathcal{E}}_k) - (|F_1(\mathcal{E}_k)| - 1)\right) && (21)\\
&\leq (|F_1(\mathcal{E}_k)| - 1)\left(\max F_1(\mathcal{E}_k) - 1 - (|F_1(\mathcal{E}_k)| - 1)\right) && (22)\\
&= (|F_1(\mathcal{E}_k)| - 1)\left(\max F_1(\mathcal{E}_k) - |F_1(\mathcal{E}_k)|\right) && (23)\\
&\leq \Phi(\mathcal{E}_k) - 1, && (24)
\end{aligned}$$

where inequality (22) follows from the fact that $\max F_1(\widetilde{\mathcal{E}}_k) \leq \max F_1(\mathcal{E}_k) - 1$, and inequality (24) follows from the fact that the set $F_1(\mathcal{E}_k)$ is not contiguous.

On the other hand, if $\mathcal{E}_{k+1}$ is specified according to Eq. (18), then we have

$$\begin{aligned}
\Phi(\mathcal{E}_{k+1}) &= |F_1(\mathcal{E}_k)|\left(\max F_1(\mathcal{E}_{k+1}) - |F_1(\mathcal{E}_k)|\right) && (25)\\
&\leq |F_1(\mathcal{E}_k)|\left(\max F_1(\mathcal{E}_k) - 1 - |F_1(\mathcal{E}_k)|\right) && (26)\\
&= \Phi(\mathcal{E}_k) - |F_1(\mathcal{E}_k)| \leq \Phi(\mathcal{E}_k) - 1, && (27)
\end{aligned}$$

where Eq. (25) follows from the fact that $|F_1(\mathcal{E}_k)| = |F_1(\mathcal{E}_{k+1})|$, and inequality (26) follows from the fact that $\max F_1(\mathcal{E}_{k+1}) \leq \max F_1(\mathcal{E}_k) - 1$. To conclude the above argument, we have that if $\Phi(\mathcal{E}_k) > 0$, then

$$\Phi(\mathcal{E}_k) - \Phi(\mathcal{E}_{k+1}) \geq 1.$$

It is straightforward to show that $\Phi(\mathcal{E}_0) < n^2$. It immediately follows that $\Phi(\mathcal{E}_{n^2}) = 0$. This finishes the proof of Claim (i).

Part 2.2: Proof of Claim (ii). According to the procedure we use in generating the sequence of edge sets, inequality (19) is trivially satisfied in the following two cases: (i) the set $F_1(\mathcal{E}_k)$ is contiguous; (ii) the set $F_1(\mathcal{E}_k)$ is not contiguous, but we have that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right) \le \mathrm{SW}\left(q^{\mathrm{NE}}(\widetilde{\mathcal{E}}_k),\, C\right)$. For the remainder of this part, we show that inequality (19) is satisfied if $\mathcal{E}_{k+1}$ is specified according to Eq. (18).

To show this, we first introduce some notation that is pertinent to the remainder of the proof. Let $n_k = |F_1(\mathcal{E}_k)|$, and assume without loss of generality that $F_1(\mathcal{E}_k) = \{i_1, \dots, i_{n_k}\}$, where $i_1 \le \cdots \le i_{n_k}$. Let $h_k = \min\left(F \setminus F_1(\mathcal{E}_k)\right)$. Since the set $F_1(\mathcal{E}_k)$ is not contiguous, we must have that $h_k < i_{n_k}$. We define a new cost function profile $C^\theta = (C_1^\theta, \dots, C_n^\theta) \in \mathcal{L}^n(c_{\min}, c_{\max})$ according to

$$C_i^\theta(s_i) = \begin{cases} (c_i s_i)^+ & \text{if } i \neq i_{n_k} \\ (\theta s_i)^+ & \text{if } i = i_{n_k}, \end{cases} \tag{28}$$

where θ is a scalar parameter to this cost function profile.

We emphasize that the unique Nash equillibrium of the networked Cournot game depends, in general, on firms' cost function profiles. We make such dependency apparent by introducing a new notation for the supply profile at Nash equilibrium. With a slight abuse of notation, for an edge set $\widetilde{\mathcal{E}} \subseteq F \times \{1\}$, and a cost function profile $\widetilde{C}$, we denote by $q^{\mathrm{NE}}(\widetilde{\mathcal{E}}, \widetilde{C})$ the unique Nash equilibrium of the corresponding game $(F, \mathcal{Q}(\widetilde{\mathcal{E}}), \widetilde{\pi})$ for the remainder of this proof.

Of interest is the monotonicity of $\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta), C^\theta)$ in the scalar θ. We have the following lemma.

Lemma 12. *Let the edge set $\widetilde{\mathcal{E}}_k$ and the cost function profile C^θ be specified according to Eq.* (17) *and* (28)*, respectively. If $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C),\, C\right) > \mathrm{SW}\left(q^{\mathrm{NE}}(\widetilde{\mathcal{E}}_k, C),\, C\right)$, then $\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta), C^\theta)$ is monotonically decreasing in θ for $c_{\min} \le \theta \le c_{i_{n_k}}$.*

The proof of Lemma 12 is deferred to Appendix I. With Lemma 12, we readily have the proof of Claim (ii). First note that if $\mathcal{E}_{k+1}$ is specified according to Eq. (18), then $\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k, C), C)$ and $\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_{k+1}, C), C)$ are related according to

$$\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k, C), C) = \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k, C^{c_{i_{n_k}}}), C^{c_{i_{n_k}}}), \tag{29}$$
$$\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_{k+1}, C), C) = \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k, C^{c_{h_k}}), C^{c_{h_k}}). \tag{30}$$

Recall that $h_k < i_{n_k}$, which implies that $c_{h_k} \le c_{i_{n_k}}$. It follows from the monotonicity of $\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta), C^\theta)$ in θ that

$$\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k, C^{c_{i_{n_k}}}), C^{c_{i_{n_k}}}) \le \mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}_k, C^{c_{h_k}}), C^{c_{h_k}}).$$

The above inequality, in combination with Eq. (29) and (30), shows that when $\mathcal{E}_{k+1}$ is specified accordding to Eq. (18), inequality (19) is still satisfied. This completes the proof of Part 2.2.

F. Proof of Theorem 8

Without loss of generality, we assume that $c_1 \le \cdots \le c_n$ According to the description of the greedy algorithm, we have that for each market $j \in M$, $(1, j) \in \mathcal{E}^*$ if and only if $c_1 < \alpha_j$. It immediately follows that

$$\mathrm{SW}^*(\mathcal{E}^*, C) = \sum_{j=1}^{m} \frac{\left((\alpha_j - c_1)^+\right)^2}{2\beta_j} = \mathrm{SW}^*(\mathcal{K}_{F,M}, C).$$

For the second part of the theorem, we only provide a proof for the case in which the number of markets $m = 1$. In this case, $M = \{1\}$. The generalization to the case in which $m > 1$ can be carried out following similar steps as in the proof of Proposition 6.

Given the restriction that $c_1 \le \cdots \le c_n$, we define the subset of ordered linear cost function profiles in $\mathcal{L}^n(c_{\min}, c_{\max})$ according to

$$\mathcal{O}^n(c_{\min}, c_{\max}) = \Big\{ C \in \mathcal{L}^n(c_{\min}, c_{\max}) \,\Big|\, C_i(s_i) = (c_i s_i)^+,\ i = 1, \dots, n,\ c_1 \le \cdots \le c_n \Big\}.$$

Thus, we have the following chain of inequalities

$$\rho(\mathcal{E}^*, C) = \inf_{\mathcal{E} \subseteq F \times M} \frac{\mathrm{SW}^*(\mathcal{K}_{F,M}, C)}{\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}), C)} \tag{31}$$
$$\le \sup_{C \in \mathcal{L}^n(c_{\min}, c_{\max})} \inf_{\mathcal{E} \subseteq F \times M} \frac{\mathrm{SW}^*(\mathcal{K}_{F,M}, C)}{\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}), C)} \tag{32}$$
$$= \sup_{C \in \mathcal{O}^n(c_{\min}, c_{\max})} \inf_{\{(1,1)\} \subseteq \mathcal{E} \subseteq F \times M} \frac{\mathrm{SW}^*(\mathcal{K}_{F,M}, C)}{\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}), C)} \tag{33}$$
$$\le \inf_{\{(1,1)\} \subseteq \mathcal{E} \subseteq F \times M} \sup_{C \in \mathcal{O}^n(c_{\min}, c_{\max})} \frac{\mathrm{SW}^*(\mathcal{K}_{F,M}, C)}{\mathrm{SW}(q^{\mathrm{NE}}(\mathcal{E}), C)} \tag{34}$$
$$\le \inf_{\{(1,1)\} \subseteq \mathcal{E} \subseteq F \times M} \frac{1}{\frac{2|\mathcal{E}|+4}{3|\mathcal{E}+5|} + \delta(\gamma_1, |\mathcal{E}|)} \tag{35}$$
$$= \frac{1}{\max\limits_{k \in \{1, \dots, n\}} \left\{ \frac{2k+4}{3k+5} + \delta(\gamma_1, k) \right\}}. \tag{36}$$

Here, Eq. (31) follows from the fact that $\mathrm{SW}^*(\mathcal{E}^*, C) = \mathrm{SW}^*(\mathcal{K}_{F,M}, C)$, Eq. (33) follows from our restriction that $c_1 \le \cdots \le c_n$, inequality (34) follows from the min-max inequality, and inequality (35) is a direct application of Proposition 6. This completes the proof.

G. Proof of Lemma 10

The proof proceeds in two steps. In Step 1, we provide a necessary and sufficient condition for the production quantity of firm k to be strictly positive at Nash equilibrium, when the edge set is given by $\mathcal{E}_k$. In Step 2, we leverage on the intermediary result in Step 1 to show that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right) > \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}),\, C\right)$ if and only if inequality (15) is satisfied.

Step 1: We show that for the game $(F, \mathcal{Q}(\mathcal{E}_k), \pi)$, the production quantity of firm k at Nash equilibrium $q_{k1}^{\mathrm{NE}}(\mathcal{E}_k)$ is strictly positive, if and only if

$$\alpha_1 - c_k > \frac{1}{k}\left(\sum_{i=1}^{k-1}(\alpha_1 - c_i)\right). \tag{37}$$

We first prove the "if" part of the above statement. First note that inequality (37) implies that the following inequality is satisfied:

$$(k+1)(\alpha_1 - c_k) > \sum_{i=1}^{k}(\alpha_1 - c_i). \tag{38}$$

One can check that if inequality (37) is satisfied, then the unique Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{E}_k), \pi)$ is given by

$$q_{i1}^{\mathrm{NE}}(\mathcal{E}_k) = \frac{(k+1)(\alpha_1 - c_i) - \sum_{\ell=1}^{k}(\alpha_1 - c_\ell)}{(k+1)\beta_1},\ i = 1, \ldots, k,$$

and $q_{i1}^{\mathrm{NE}}(\mathcal{E}_k) = 0$ for $i = k+1, \ldots, n$. It follows from inequality (38) that $q_{k1}^{\mathrm{NE}}(\mathcal{E}_k) > 0$.

Next, we prove the "only if" part of the above statement. By assumption of this lemma, we have that $c_i \leq c_k$ for each $i = 1, \ldots, k$. Recall that for the game $(F, \mathcal{Q}(\mathcal{E}_k), \pi)$, firm k's production quantity $q_{k1}^{\mathrm{NE}}(\mathcal{E}_k)$ at Nash equilibrium is strictly positive. It follows that $q_{i1}^{\mathrm{NE}}(\mathcal{E}_k) > 0$ for $i = 1, \ldots, k$. The first order optimality condition for Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{E}_k), \pi)$ implies that

$$\alpha_1 - \beta_1\left(\sum_{\ell=1}^{k} q_{\ell 1}^{\mathrm{NE}}(\mathcal{E}_k)\right) - \beta_1 q_{i1}^{\mathrm{NE}}(\mathcal{E}_k) - c_i = 0, \tag{39}$$

for $i = 1, \ldots, k$. Consequently, we have that

$$q_{k1}^{\mathrm{NE}}(\mathcal{E}_k) = \frac{(k+1)(\alpha_1 - c_k) - \sum_{\ell=1}^{k}(\alpha_1 - c_\ell)}{(k+1)\beta_1} > 0.$$

This immediately implies that inequality (37) is satisfied.

Step 2: We show $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right) > \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}),\, C\right)$ if and only if inequality (15) is satisfied. First note when $k = 1$, it is straightforward to see that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_1),\, C\right) > 0$ if and only if $\alpha_1 - c_1 > 0$. Thus, for the remainder of the proof, we assume that $k \geq 2$.

We only provide the "only if" part of the proof, as the "if" part of the proof follows from analogous arguments. First note that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k),\, C\right) > \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}),\, C\right)$ implies that $q_{k1}^{\mathrm{NE}}(\mathcal{E}_k) > 0$. If this is not the case, then we have that $q^{\mathrm{NE}}(\mathcal{E}_k) = q^{\mathrm{NE}}(\mathcal{E}_{k-1})$, which clearly leads to a contradiction. Since $c_i \leq c_k$ for all $i = 1, \ldots, k$, we have that

$$q_{i1}^{\mathrm{NE}}(\mathcal{E}_k) \geq q_{k1}^{\mathrm{NE}}(\mathcal{E}_k) > 0, \quad i = 1, \ldots, k.$$

One can show that the unique Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{E}_k), \pi)$ is given by

$$q_{i1}^{\mathrm{NE}}(\mathcal{E}_k) = \frac{(k+1)(\alpha_1 - c_i) - \sum_{\ell=1}^{k}(\alpha_1 - c_\ell)}{(k+1)\beta_1},\ i = 1, \ldots, k,$$

and $q_{i1}^{\mathrm{NE}}(\mathcal{E}_k) = 0$ for $i = k+1, \ldots, n$. The aggregate supply in market 1 at the unique Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{E}_k), \pi)$ is given by

$$d_1^{\mathrm{NE}}(\mathcal{E}_k) = \sum_{i=1}^{n} q_{i1}^{\mathrm{NE}}(\mathcal{E}_k) = \frac{\sum_{i=1}^{k}(\alpha_1 - c_i)}{(k+1)\beta_1}$$

Additionally, the social welfare at the Nash equilibrium of the game $(F, \mathcal{Q}(\mathcal{E}_k), \pi)$ satisfies

$$\begin{aligned}
&\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C\right)\\
=&\alpha_1 d_1^{\mathrm{NE}}(\mathcal{E}_k) - \frac{1}{2}\beta_1 d_1^{\mathrm{NE}^2}(\mathcal{E}_k) - \sum_{i=1}^{k} c_i q_{i1}^{\mathrm{NE}}(\mathcal{E}_k)\\
=&\alpha_1 d_1^{\mathrm{NE}}(\mathcal{E}_k) - \frac{1}{2}\beta_1 d_1^{\mathrm{NE}^2}(\mathcal{E}_k) - \sum_{i=1}^{k} c_i\left(\frac{\alpha_1 - c_i}{\beta_1} - d_1^{\mathrm{NE}}(\mathcal{E}_k)\right)\\
=&(k+1)\alpha_1 d_1^{\mathrm{NE}}(\mathcal{E}_k) - \frac{1}{2}\beta_1 d_1^{\mathrm{NE}^2}(\mathcal{E}_k) - \sum_{i=1}^{k}\left(\frac{c_i(\alpha_1 - c_i)}{\beta_1}\right)\\
&+\left(-k\alpha_1 + \sum_{i=1}^{k} c_i\right) d_1^{\mathrm{NE}}(\mathcal{E}_k)\\
=&\frac{\alpha_1\sum_{i=1}^{k}(\alpha_1 - c_i)}{\beta_1} - \frac{2k+3}{2}\beta_1 d_1^{\mathrm{NE}^2}(\mathcal{E}_k) - \sum_{i=1}^{k}\frac{c_i(\alpha_1 - c_i)}{\beta_1}\\
=&\frac{\sum_{i=1}^{k}(\alpha_1 - c_i)^2}{\beta_1} - \frac{2k+3}{2}\beta_1 d_1^{\mathrm{NE}^2}(\mathcal{E}_k)\\
=&\frac{\sum_{i=1}^{k}(\alpha_1 - c_i)^2}{\beta_1} - \frac{2k+3}{2}\frac{\left(\sum_{i=1}^{k}(\alpha_1 - c_i)\right)^2}{(k+1)^2\beta_1}.
\end{aligned}$$

Next, we provide a closed-form expression for $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}), C\right)$. Recall that $q_{k1}^{\mathrm{NE}}(\mathcal{E}_k) > 0$. According to our result in Step 1, inequality (37) is satisfied. This implies that

$$\alpha_1 - c_{k-1} \geq \alpha_1 - c_k > \frac{1}{k}\left(\sum_{i=1}^{k-1}(\alpha_1 - c_i)\right),$$

which further implies that

$$\alpha_1 - c_{k-1} > \frac{1}{k-1}\left(\sum_{i=1}^{k-2}(\alpha_1 - c_i)\right).$$

According to our result in Step 1, it follows that for the game $(F, \mathcal{Q}(\mathcal{E}_{k-1}), \pi)$, producer $k-1$'s production quantity at Nash equilibrium is strictly positive. Using similar arguments as in our derivation on the closed-form expression for $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C\right)$, we have the following closed-form expression for $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}), C\right)$

$$\begin{aligned}
&\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}), C\right)\\
=&\frac{\sum_{i=1}^{k-1}(\alpha_1 - c_i)^2}{\beta_1} - \frac{2k+1}{2}\frac{\left(\sum_{i=1}^{k-1}(\alpha_1 - c_i)\right)^2}{k^2\beta_1}
\end{aligned}$$

Thus, the difference between $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C\right)$ and $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}), C\right)$ is given by

$$\begin{aligned}&\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C\right) - \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}), C\right)\\ =&\frac{(\alpha_1 - c_k)^2}{\beta_1} + \frac{2k+1}{2}\frac{\left(\sum_{i=1}^{k-1}(\alpha_1 - c_i)\right)^2}{k^2\beta_1}\\ &- \frac{2k+3}{2}\frac{\left(\sum_{i=1}^{k}(\alpha_1 - c_i)\right)^2}{(k+1)^2\beta_1}.\end{aligned}$$

An algebraic calculation reveals that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C\right) > \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}), C\right)$ if and only if

$$\frac{\alpha_1 - c_k}{\sum_{i=1}^{k-1}(\alpha_1 - c_i)} < \frac{1}{k}, \quad \text{or} \quad \frac{\alpha_1 - c_k}{\sum_{i=1}^{k-1}(\alpha_1 - c_i)} > \frac{k+2+\frac{1}{2k}}{k^2+k-\frac{1}{2}}.$$

Recall that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C\right) > \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}), C\right)$ implies that inequality (37) is satisfied. It follows from the above inequality that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C\right) > \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_{k-1}), C\right)$ implies inequality (15) is satisfied.

H. Proof of Lemma 11

We prove this lemma by induction. For the base step $k = k^* + 1$, inequality (16) is satisfied by the assumption of this lemma.

Assume that inequality (16) is satisfied for k, where $k \geq 1$. We now show that it is satisfied for $k+1$. That is, we show that

$$\alpha_1 - c_{k+1} \leq \frac{1}{k+1}\left(1 + \frac{1}{k+1-\frac{1}{2(k+2)}}\right)\left(\sum_{i=1}^{k}(\alpha_1 - c_i)\right). \tag{40}$$

Since $c_{k+1} \geq c_k$, inequality (40) is satisfied if the following inequality holds

$$(k+1)(\alpha_1 - c_k) \leq \left(1 + \frac{1}{k+1-\frac{1}{2(k+2)}}\right)\left(\sum_{i=1}^{k}(\alpha_1 - c_i)\right). \tag{41}$$

An algebraic calculation reveals that inequality (41) is satisfied, if and only if

$$\frac{k^3+3k^2+\frac{1}{2}k-2}{k^2+3k+\frac{3}{2}}(\alpha_1 - c_k) \leq \frac{k^2+4k+\frac{7}{2}}{k^2+3k+\frac{3}{2}}\left(\sum_{i=1}^{k-1}(\alpha_1 - c_i)\right).$$

Under the assumption that $k \geq 1$, the above inequality is satisfied, if and only if

$$\alpha_1 - c_k \leq \frac{k^2+4k+\frac{7}{2}}{k^3+3k^2+\frac{1}{2}k-2}\left(\sum_{i=1}^{k-1}(\alpha_1 - c_i)\right). \tag{42}$$

The inductive assumption implies that inequality (16) is satisfied for k. Given that $c_i \leq \alpha_1$ for $i = 1, \ldots, n$, we have that inequality (42) is satisfied if

$$\frac{1}{k}\left(1 + \frac{1}{k - \frac{1}{2(k+1)}}\right) \leq \frac{k^2+4k+\frac{7}{2}}{k^3+3k^2+\frac{1}{2}k-2}. \tag{43}$$

Given that $k \geq 1$, inequality (43) holds if and only if

$$\begin{aligned}&\left(k^2+2k+\frac{1}{2}\right)\left(k^3+3k^2+\frac{1}{2}k-2\right)\\ &\leq k\left(k^2+k-\frac{1}{2}\right)\left(k^2+4k+\frac{7}{2}\right).\end{aligned}$$

And the above inequality holds if and only if $(k+1)^2 \geq 0$. Thus, (43) is satisfied if $k \geq 1$. This further implies that inequalities (40)-(42) are all satisfied. Hence, inequality (16) also holds for $k+1$. This completes the proof by induction.

I. Proof of Lemma 12

The proof proceeds in two steps. In Step 1, we provide a closed-form expression for $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta), C^\theta\right)$, and show that it is piecewise quadratic in θ. In Step 2, we show that given the assumption stated in this lemma, $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta), C^\theta\right)$ is strictly decreasing in θ for $c_{\min} \leq \theta \leq c_{i_{n_k}}$.

Step 1: We first show that when $\theta = c_{i_{n_k}}$, we have that $q_{i1}^{\mathrm{NE}}(\mathcal{E}, C^\theta) > 0$ for all $i \in F_1(\mathcal{E}_k)$. To see this, note that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C), C\right) > \mathrm{SW}(q^{\mathrm{NE}}(\widetilde{\mathcal{E}}_k, C), C)$. This implies that $q_{i_{n_k}1}^{\mathrm{NE}}(\mathcal{E}_k, C) > 0$. If this is not the case, we must have that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C), C\right) = \mathrm{SW}(q^{\mathrm{NE}}(\widetilde{\mathcal{E}}_k, C), C)$, which is a contradiction. Using similar arguments as in the proof of Lemma 10, it is straightforward to show that if $\theta = c_{i_{n_k}}$, $q_{i1}^{\mathrm{NE}}(\mathcal{E}, C^\theta) > 0$ for all $i \in F_1(\mathcal{E}_k)$.

Assume that when $\theta = c_{\min}$, the vector $q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta)$ includes $n_{\min}$ strictly positive entries. Since $|\mathcal{E}_k| = n_k$, we must have that $n_{\min} \leq n_k$. We define a collection of subsets $\Theta_{n_{\min}}, \ldots, \Theta_{n_k}$ of the set $[c_{\min}, c_{i_{n_k}}]$ as follows:

$$\begin{aligned}\Theta_\ell = &\left[\alpha_1 + \sum_{r=1}^{\ell-2}(\alpha_1 - c_{i_r}) - \ell(\alpha_1 - c_{i_{\ell-1}}),\right.\\ &\left.\alpha_1 + \sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r}) - (\ell+1)(\alpha_1 - c_{i_\ell})\right) \bigcap \left[c_{\min}, c_{i_{n_k}}\right],\end{aligned}$$

where $\ell = n_{\min}, \ldots, n_k$. Given that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C), C\right) > \mathrm{SW}(q^{\mathrm{NE}}(\widetilde{\mathcal{E}}_k, C), C)$, one can show that

$$\bigcup_{\ell=n_{\min}}^{n_k}\Theta_\ell = \left[c_{\min}, c_{i_{n_k}}\right], \quad \text{and} \quad \Theta_{\ell_1}\bigcap\Theta_{\ell_2} = \emptyset$$

for any $\ell_1, \ell_2 \in \{n_{\min}, \ldots, n_k\}$ satisfying $\ell_1 \neq \ell_2$. [3] Additionally, for any $\theta \in \mathrm{int}(\Theta_\ell)$, $\ell \in \{n_{\min}, \ldots, n_k\}$, we have

$$q_{i1}^{\mathrm{NE}}(\mathcal{E}_k, C^\theta) > 0, \text{ for } i = i_1, \ldots, i_{\ell-1}, \text{ and } i = i_{n_k}.$$

That is, the vector $q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta)$ contains ℓ strictly positive entries. Using similar arguments as in the proof of Lemma

[3] The collection of sets $\{\Theta_{n_{\min}}, \ldots, \Theta_{n_k}\}$ might not be a partition of $[c_{\min}, c_{i_{n_k}}]$, as some of these sets can be empty.

10, one can show that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta), C^\theta\right)$ admits the following closed-form expression

$$\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta), C^\theta\right) = \frac{(\alpha_1-\theta)^2 + \sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r})^2}{\beta_1} - \frac{2\ell+3}{2}\frac{\left(\alpha_1 - \theta + \sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r})\right)^2}{(\ell+1)^2\beta_1}, \quad \text{if } \theta \in \Theta_\ell.$$

We remark that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta), C^\theta\right)$ is a piecewise quadratic function of θ that is continuous in θ for $\theta \in [c_{\min}, c_{i_{n_k}}]$, and continuously differentiable in θ for $\theta \in \mathrm{int}(\Theta_\ell)$, $\ell \in \{n_{\min}, \ldots, n_k\}$.

Step 2: We show that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C^\theta), C^\theta\right)$ is strictly monotonically decreasing in θ for $\theta \in [c_{\min}, c_{i_{n_k}}]$. In showing this, we first show that $\partial\, \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C^\theta\right)/\partial\theta < 0$ for $\theta \in \mathrm{int}(\Theta_\ell)$ for $\ell = n_{\min}, \ldots, n_k$.

Fix $\ell \in \{n_{\min}, \ldots, n_k\}$. For $\theta \in \mathrm{int}(\Theta_\ell)$, we have the following closed-form expression for $\partial\, \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C^\theta\right)/\partial\theta$:

$$\begin{aligned}&\frac{\partial}{\partial\theta}\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C^\theta\right)\\ =&\frac{(2\ell+3)\sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r}) - (2\ell^2+2\ell-1)(\alpha_1-\theta)}{(\ell+1)^2\beta_1}.\end{aligned}$$

Thus, $\partial\, \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C^\theta\right)/\partial\theta < 0$ if the following inequality is satisfied:

$$\alpha_1 - \theta > \frac{2\ell+3}{2\ell^2+2\ell-1}\sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r}). \tag{44}$$

Recall that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k, C), C\right) > \mathrm{SW}(q^{\mathrm{NE}}(\widetilde{\mathcal{E}}_k, C), C)$. It follows from Lemma 10 that the following inequality is satisfied

$$\alpha_1 - c_{i_{n_k}} > \frac{1}{n_k}\left(1 + \frac{1}{n_k - \frac{1}{2(n_k+1)}}\right)\sum_{r=1}^{n_k-1}(\alpha_1 - c_{i_r}). \tag{45}$$

Since $q^{\mathrm{NE}}_{i_r 1}(\mathcal{E}_k, C) > 0$ for $r = 1, \ldots, n_k$, we clearly have that $c_{i_r} < \alpha_1$ for $r = 1, \ldots, n_k$. By Lemma 11, inequality (45) implies that

$$\alpha_1 - c_{i_\ell} > \frac{1}{\ell}\left(1 + \frac{1}{\ell - \frac{1}{2(\ell+1)}}\right)\sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r}) \tag{46}$$

for $\ell = n_{\min}, \ldots, n_k$. Inequality (46) implies that

$$\begin{aligned}\ell\,(\alpha_1 - c_{i_\ell}) &> \left(1 + \frac{1}{\ell - \frac{1}{2(\ell+1)}}\right)\sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r})\\ &> \sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r})\end{aligned} \tag{47}$$

It follows from inequality (47) that the following chain of inequalities are satisfied for $\theta \in \mathrm{int}(\Theta_\ell)$:

$$\begin{aligned}\theta <&\alpha_1 + \sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r}) - (\ell+1)(\alpha_1 - c_{i_\ell})\\ =&c_{i_\ell} + \sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r}) - \ell(\alpha_1 - c_{i_\ell}) < c_{i_\ell}.\end{aligned}$$

The above inequality, in combination with inequality (46), provides the following lower bound on $\alpha_1 - \theta$ when $\theta \in \mathrm{int}(\Theta_\ell)$:

$$\begin{aligned}\alpha_1 - \theta >&\alpha_1 - c_{i_\ell} > \frac{1}{\ell}\left(1 + \frac{1}{\ell - \frac{1}{2(\ell+1)}}\right)\sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r})\\ =&\frac{1}{\ell}\left(1 + \frac{\ell+1}{\ell^2+\ell-\frac{1}{2}}\right)\sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r})\\ >&\frac{1}{\ell}\left(1 + \frac{\ell+1}{2\ell^2+2\ell-1}\right)\sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r})\\ =&\frac{2\ell+3}{2\ell^2+2\ell-1}\sum_{r=1}^{\ell-1}(\alpha_1 - c_{i_r})\end{aligned}$$

Thus, $\partial\, \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C^\theta\right)/\partial\theta < 0$ for $\theta \in \mathrm{int}(\Theta_\ell)$.

Recall that the union of the intervals $\Theta_{n_{\min}}, \ldots, \Theta_{n_k}$ satisfies

$$\bigcup_{\ell=n_{\min}}^{n_k}\Theta_\ell = \left[c_{\min}, c_{i_{n_k}}\right].$$

Additionally, $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C^\theta\right)$ is continuous in θ for $\theta \in [c_{\min}, c_{i_{n_k}}]$. Consequently, the fact $\partial\, \mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C^\theta\right)/\partial\theta < 0$ for $\theta \in \mathrm{int}(\Theta_\ell)$ for each $\ell = n_{\min}, \ldots, n_k$ implies that $\mathrm{SW}\left(q^{\mathrm{NE}}(\mathcal{E}_k), C^\theta\right)$ is monotonically strictly decreasing in θ on $[c_{\min}, c_{i_{n_k}}]$.