Decentralized Control of Distributed Energy Resources in Radial Distribution Systems

Weixuan Lin Eilyan Bitar

Abstract—**We consider the decentralized control of radial distribution systems with controllable photovoltaic inverters and storage devices. For such systems, we consider the problem of designing controllers that minimize the expected cost of meeting demand, while respecting distribution system and resource constraints. Employing a linear approximation of the branch flow model, we formulate this problem as the design of a decentralized disturbance-feedback controller that minimizes the expected value of a convex quadratic cost function, subject to convex quadratic constraints on the state and input. As such problems are, in general, computationally intractable, we derive an inner approximation to this decentralized control problem, which enables the efficient computation of an affine control policy via the solution of a conic program. As affine policies are, in general, suboptimal for the systems considered, we provide an efficient method to bound their suboptimality via the solution of another conic program. A case study of a 12 kV radial distribution feeder demonstrates that decentralized affine controllers can perform close to optimal.**

I. INTROUDCTION

The increasing penetration of distributed and renewable energy resources introduces challenges to the distribution system, including rapid fluctuations in bus voltage magnitudes, reverse power flows at substations, and deteriorated power quality, due to the intermittency of supply from renewables. However, traditional techniques for distribution system management, including the deployment of on-load tap changing (OLTC) transformers and the installation of shunt capacitors, cannot effectively deal with the rapid variation in power supply from renewable resources [1]. In this paper, we aim to address such challenges by developing a systematic approach to the design of decentralized controllers for networks with a high penetration of distributed solar and energy storage resources, in order to minimize the expected cost of meeting demand, while respecting network and resource constraints.

Related Work: Although current industry standards require that photovoltaic (PV) inverters operate at a unity power factor [2], the latent reactive power capacity of PV inverters can be utilized to regulate voltage profiles [3]–[8] and reduce active power losses [9]–[15] in distribution networks. A large fraction of the litearture on the reactive power management of PV inverters seeks to solve an optimal power flow (OPF) problem [3]–[13], whose solution determines the reactive power injections of PV inverters in real time. Because of the fast changes in demand and active power supply from PV inverters, such OPF problems are solved repeatedly on a fast time scale (e.g., every minute). In the presence of a large number of PV inverters, the sheer size of the resulting OPF problem that needs to be solved, and the communication requirements it entails, give rise to the need for distributed optimization methods [5]–[12]. Additionally, several recent papers have attempted to explicitly treat uncertainty in renewable supply and demand by leveraging on methods grounded in stochastic optimization [14], [15].

Apart from controlling the reactive power injection from PV inverters, one can manage the sequence of power injections of distributed storage devices (e.g. electric vehicles or battery storage devices) to mitigate voltage fluctuations or reduce distribution losses [1], [16]–[19]. Since control inputs to storage devices are coupled across time, the problem of managing their charging profile amounts to a multi-period stochastic control problem. In the presence of network constraints and uncertainty in demand and renewable supply, the calculation of the optimal control policy is, in general, computationally intractable. The development of computational methods to enable the tractable calculation of feasible control policies with computable bounds on their suboptimality is therefore desired.

Contribution: The setting we consider entails the decentralized control of distributed energy resources spread throughout a radial distribution network, subject to uncertainty in demand and renewable supply. Our objective is to minimize the expected amount of active power supplied at the substation to meet the demand, subject to network and resource constraints. Such an objective function is equivalent to minimizing the sum of expected active power losses and terminal storage states. The determination of an optimal decentralized control policy in such a setting is, in general, intractable. Our primary contributions are two-fold. First, we develop a convex programming approach to the design of decentralized, affine disturbance-feedback controllers. Second, as such control policies are, in general, suboptimal, we provide a technique to bound their suboptimality through the solution of another convex program. We verify that the decentralized affine policies we derive are close to optimal for the family of problem instances considered in our case study.

Organization: This paper is organized as follows. Section II introduces the distribution network and energy resource models. Section III formally states the decentralized control design problem. Section IV describes an approach to the computation of decentralized affine control policies via convex programming. Section V describes an approach to the computation of bounds on the suboptimality incurred by these affine control policies via convex programming. Section VI presents a numerical study of a 12 kV radial distribution feeder.

Supported in part by NSF grant ECCS-1351621, NSF grant CNS-1239178, NSF grant IIP-1632124, PSERC under sub-award S-52, and US DoE under the CERTS initiative.

Weixuan Lin (wl476@cornell.edu) and Eilyan Bitar (eyb5@cornell.edu) are with the School of Electrical and Computer Engineering, Cornell University, Ithaca, NY, 14853, USA.

Notation: Let $\mathbf{R}$ denote the set of real numbers. Denote the transpose of vector $x \in \mathbf{R}^n$ by x'. Given a process $\{x(t)\}$ indexed by $t = 0, \ldots, T-1$, we denote by $x^t = (x(0), \ldots, x(t))$ its history up until and including time t. We denote the trace of a square matrix A by $\mathrm{Tr}\,(A)$. We denote by $\mathcal{K}$ a proper cone (i.e., convex, closed and pointed with an nonempty interior). Let $\mathcal{K}^*$ denote its dual cone. For a matrix A of appropriate dimension, $A \succeq_{\mathcal{K}} 0$ denotes its columnwise inclusion in $\mathcal{K}$.

II. Network and Resource Models

A. Branch Flow Model and its Linear Approximation

Consider a radial distribution network whose topology is described by a *rooted tree* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{0, 1, .., n\}$ denotes its set of (nodes) buses, and $\mathcal{E}$ its set of (directed edges) distribution lines. In particular, bus 0 is defined as the root of the network, and represents the substation that connects to the external power system. Each directed distribution line admits the natural orientation, i.e., away from the root. For each distribution line $(i, j) \in \mathcal{E}$, we denote by $r_{ij} + \mathbf{i}x_{ij}$ its *impedance*. In addition, define I_{ij} as the *complex current* flowing from bus i to j, and $p_{ij} + \mathbf{i}q_{ij}$ as the *complex power* flowing from bus i to j. For each bus $i \in \mathcal{V}$, V_i denotes its *complex voltage*, and $p_i + \mathbf{i}q_i$ is the *complex power injection* at said bus. We assume that the complex voltage V_0 at the substation is fixed and known.

We employ the *branch flow model* proposed in [20], [21] to describe the steady-state, single-phase AC power flow equations associated with this radial distribution network. For each bus $j = 1, \ldots, n$, and its unique *parent* $i \in \mathcal{V}$, we have

$$-p_j \;=\; p_{ij} - r_{ij}\ell_{ij} - \sum_{k:(j,k)\in\mathcal{E}} p_{jk}, \tag{1}$$

$$-q_j \;=\; q_{ij} - x_{ij}\ell_{ij} - \sum_{k:(j,k)\in\mathcal{E}} q_{jk}, \tag{2}$$

$$v_j^2 \;=\; v_i^2 - 2(r_{ij}p_{ij} + x_{ij}q_{ij}) + (r_{ij}^2 + x_{ij}^2)\ell_{ij}, \tag{3}$$

$$\ell_{ij} \;=\; (p_{ij}^2 + q_{ij}^2)/v_i^2, \tag{4}$$

where $\ell_{ij} = |I_{ij}|^2$ and $v_i = |V_i|$. We note that the branch flow model is well defined only for radial distribution networks, as we require that each bus j (excluding the substation) have a unique parent $i \in \mathcal{V}$.

For the remainder of the paper, we consider a linear approximation of the branch flow model (1)-(4) based on the Simplified Distflow method in [22]. To derive this approximation, we assume that $\ell_{ij} = 0$ for all $(i, j) \in \mathcal{E}$, as the active and reactive power losses on distribution lines are considered small relative to the power flows. According to [4], [23], such an approximation tends to introduces a relative model error of 1-5% for practical distribution networks. Under this assumption, Eq. (1)-(3) can be reformulated as

$$-p_j \;=\; p_{ij} - \sum_{k:(j,k)\in\mathcal{E}} p_{jk}, \tag{5}$$

$$-q_j \;=\; q_{ij} - \sum_{k:(j,k)\in\mathcal{E}} q_{jk}, \tag{6}$$

$$v_j^2 \;=\; v_i^2 - 2(r_{ij}p_{ij} + x_{ij}q_{ij}). \tag{7}$$

The linearized branch flow Eq. (5)-(7) can be written more compactly as

$$v^2 = Rp + Xq + v_0^2\mathbf{1}. \tag{8}$$

Here, $v^2 = (v_1^2, .., v_n^2)$, $p = (p_1, .., p_n)$, and $q = (q_1, .., q_n)$ denote the vectors of squared bus voltage magnitudes, real power injections, and reactive power injections, respectively, and $\mathbf{1} = (1, .., 1)$ is a vector of all ones in $\mathbf{R}^n$. The matrices $R, X \in \mathbf{R}^{n\times n}$ are defined according to

$$R_{ij} = 2\sum_{(h,k)\in\mathcal{P}_i\cap\mathcal{P}_j} r_{hk},$$

$$X_{ij} = 2\sum_{(h,k)\in\mathcal{P}_i\cap\mathcal{P}_j} x_{hk},$$

where $\mathcal{P}_i \subset \mathcal{E}$ is defined as the set of edges on the unique path from bus 0 to i.

For the remainder of this paper, we consider the control of photovoltaic (PV) inverters and storage devices in the distribution network over discrete time periods indexed by $t = 0, .., T-1$. We require the vector of voltage magnitudes $v(t) = (v_1(t), .., v_n(t)) \in \mathbf{R}^n$ at each time t to satisfy

$$\underline{v} \le v(t) \le \overline{v}, \tag{9}$$

where the allowable range of voltage magnitudes is defined by $\underline{v}, \overline{v} \in \mathbf{R}^n$.

B. Energy Storage Model

We consider a distribution network consisting of n perfectly efficient energy storage devices, where each bus i (excluding the substation) is assumed to have an energy storage capacity of $b_i \in \mathbf{R}$. The dynamics of storage device i is described by

$$x_i(t+1) = x_i(t) - p_i^S(t), \quad t = 0, \ldots, T-1, \tag{10}$$

where the state $x_i(t) \in \mathbf{R}$ denotes the amount of energy stored in storage device i just preceding period t, and $p_i^S(t) \in \mathbf{R}$ denotes the amount of energy extracted from storage device i during period t. We assume that the initial energy storage state $x_i(0)$ is fixed and known. We impose energy storage state and input constraints of the form

$$0 \le x_i(t) \le b_i, \quad t = 0, \ldots, T \tag{11}$$

$$\underline{p}_i^S \le p_i^S(t) \le \overline{p}_i^S, \quad t = 0, \ldots, T-1. \tag{12}$$

for $i = 1, .., n$. Here, $\underline{p}_i^S, \overline{p}_i^S \in \mathbf{R}$ define the range of allowable inputs for storage device i at each time period t.

C. The Photovoltaic Inverter Model

We assume that, in addition to energy storage, each bus i (excluding the substation) has a photovoltaic (PV) inverter whose reactive power injection can be controlled. For each bus $i = 1, \ldots, n$, we denote by $\xi_i^I(t) \in \mathbf{R}$ and $q_i^I(t) \in \mathbf{R}$ the active and reactive power injection from the PV inverter at time t, respectively. We model $\xi_i^I(t)$ as a discrete-time stochastic process, whose precise specification is presented in Section

II-E. Additionally, we require that the reactive power injections respect capacity constraints of the form

$$\left|q_i^I(t)\right| \leq \sqrt{{s_i^I}^2 - \xi_i^I(t)^2}, \quad i = 1, \dots, n, \tag{13}$$

for $t = 0, \dots, T-1$, where $s_i^I \in \mathbf{R}$ is the apparent power capacity of inverter i. Clearly, it must hold that $\xi_i^I(t) \leq s_i^I$.

D. The Load Model

Each bus in the distribution network is assumed to have a constant power load, which we will treat as a discrete-time stochastic process. Accordingly, we denote by $\xi_i^p(t) \in \mathbf{R}$ and $\xi_i^q(t) \in \mathbf{R}$ the active and reactive power demand, respectively, at bus i and time t. It follows that the nodal active and reactive power balance equations can be expressed as

$$p_i(t) = p_i^S(t) + \xi_i^I(t) - \xi_i^p(t), \tag{14}$$

$$q_i(t) = q_i^I(t) - \xi_i^q(t). \tag{15}$$

where $p_i(t) \in \mathbf{R}$ and $q_i(t) \in \mathbf{R}$ denote the active and reactive power injections, respectively, at each bus $i = 1, \dots, n$ and time period $t = 0, \dots, T-1$.

E. The Uncertainty Model

We model the active power injection of PV inverters, and the active and reactive power demand as discrete-time stochastic processes. Accordingly, we associate with each bus i a *disturbance process* defined as $\xi_i(t) = (\xi_i^p(t), \xi_i^q(t), \xi_i^I(t)) \in \mathbf{R}^3$. We define the *full disturbance trajectory* as

$$\xi = (1, \xi(0), \dots, \xi(T-1)) \in \mathbf{R}^{N_\xi}, \quad N_\xi = 1 + 3nT, \tag{16}$$

where $\xi(t) = (\xi_1(t), \dots, \xi_n(t)) \in \mathbf{R}^{3n}$ for each time t. Note that, in our specification of the disturbance trajectory ξ, we have included a constant scalar as its initial component. Such notational convention will prove useful in simplifying the specification of affine control policies in the sequel.

We assume that the disturbance trajectory ξ has support Ξ that is a nonempty and compact subset of $\mathbf{R}^{N_\xi}$, representable by

$$\Xi = \{\xi \in \mathbf{R}^{N_\xi} \mid \xi_1 = 1 \text{ and } W\xi \succeq_{\mathcal{K}} 0\},$$

where the matrix $W \in \mathbf{R}^{\ell \times N_\xi}$ is known. Because of the compactness of Ξ, the second-order moment matrix $M = \mathbf{E}(\xi\xi')$, is finite-valued. Without loss of generality, we further assume that M is positive definite. We emphasize that our specification of the disturbance trajectory ξ captures a large family of disturbance processes, including those whose support can be described as the intersection of polytopes and ellipsoids.

III. Decentralized Control Design

A. State Space Description

In what follows, we build on the individual resource models developed in Section II to develop a discrete-time state space model describing the collective dynamics of the distribution network. The system consists of n subsystems, where each subsystem $i = 1, .., n$ represents the collection of resources connected to bus i. For each subsystem i, we let the energy storage state $x_i(t)$ be its *state* at time t, and define its *input* according to

$$u_i(t) = \begin{bmatrix} p_i^S(t) \\ q_i^I(t) \end{bmatrix}.$$

The state equation for each subsystem i is given by Eq. (10).

We define the full system state and input at time t by $x(t) = (x_1(t), .., x_n(t)) \in \mathbf{R}^n$ and $u(t) = (u_1(t), .., u_n(t)) \in \mathbf{R}^{2n}$. The full system equation admits the following representation

$$x(t+1) = x(t) + Bu(t).$$

Here, the matrix B is given by

$$B = I_n \otimes \begin{bmatrix} -1 & 0 \end{bmatrix},$$

where $\otimes$ denotes the Kronecker product operator. The initial system state $x(0)$ is assumed fixed and known. The full system state and input trajectories, and the initial system state $x(0)$ are related according to

$$x = \mathbb{A}x(0) + \mathbb{B}u, \tag{17}$$

where x and u represent the *state* and *input trajectories*, respectively[1]. They are defined according to

$$x = (x(0), \dots, x(T)) \in \mathbf{R}^{N_x}, \qquad N_x = n(T+1),$$
$$u = (u(0), \dots, u(T-1)) \in \mathbf{R}^{N_u}, \qquad N_u = 2nT.$$

B. Decentralized Control Design

At each time t, each subsystem i must determine its local control input based on its available information. In this paper, we restrict ourselves to fully decentralized disturbance-feedback control policies. Namely, for each subsystem i, its control input at time t is restricted to be of the form

$$u_i(t) = \gamma_i(\xi_i^t, t),$$

where $\gamma_i(\cdot, t)$ is a causal measurable function of the local disturbance history. We define the *local control policy* for subsystem i as $\gamma_i = (\gamma_i(\cdot, 0), .., \gamma_i(\cdot, T-1))$. We refer to the collection of local control policies $\gamma = (\gamma_1, .., \gamma_n)$ as the *decentralized control policy*, and define Γ as the family of all admissible decentralized control policies.

Our objective is to minimize the expected cost of meeting demand over the distribution network, which we measure according to the expected amount of active power supplied at the substation. Within the context of our formulation, this is equivalent to minimizing the sum of expected active power losses and terminal storage states. In a similar spirit to [3], [22], we approximate the active power loss on line $(i, j) \in \mathcal{E}$ according to

$$\delta p_{ij}(t) = r_{ij}(p_{ij}(t)^2 + q_{ij}(t)^2)/v_0(t)^2,$$

for $t = 0, \dots, T-1$. Implicit in this approximation is the assumption that the bus voltage magnitudes are uniform across the network. By Eq. (5)-(6) and (14)-(15), one can write the total active power losses over time periods $t = 0, \dots, T-1$

[1] Due to space constraints, the matrices $(\mathbb{A}, \mathbb{B})$ are specified in the Appendix of [24].

as a quadratic function of the vector (u, ξ). Namely, one can construct matrices $L_u \in \mathbf{R}^{2nT \times N_u}$, $L_\xi \in \mathbf{R}^{2nT \times N_\xi}$, and a positive definite diagonal matrix $\Sigma \in \mathbf{R}^{2nT \times 2nT}$, such that

$$(L_u u + L_\xi \xi)' \Sigma (L_u u + L_\xi \xi) = \sum_{t=0}^{T-1} \sum_{(i,j) \in \mathcal{E}} \delta p_{ij}(t). \quad (18)$$

The sum of terminal storage states of all storage devices can be written as a linear function of the state trajectory x. Namely, one can construct a vector $c \in \mathbf{R}^{N_x}$, such that

$$c'x = \sum_{i=1}^{n} x_i(T). \quad (19)$$

Henceforth, we define the expected cost associated with a decentralized control policy $\gamma \in \Gamma$ according to

$$J(\gamma) = \mathbf{E}^\gamma \left[c'x + (L_u u + L_\xi \xi)' \Sigma (L_u u + L_\xi \xi) \right], \quad (20)$$

where expectation is taken with respect to the joint distribution on (x, u, ξ) induced by the control policy γ. [2] We define the *optimal decentralized control* problem as

$$\begin{array}{ll} \text{minimize} & J(\gamma) \\ \text{subject to} & \gamma \in \Gamma \\ & \left. \begin{array}{l} x = \mathbb{A}x(0) + \mathbb{B}u \\ u = \gamma(\xi) \\ x \in \mathcal{X}, \ u \in \mathcal{U}(\xi) \end{array} \right\} \forall \xi \in \Xi, \end{array} \quad (21)$$

where the set of feasible states $\mathcal{X}$ is defined by inequality (11), and the set of feasible control inputs $\mathcal{U}(\xi)$ is defined according to inequalities (9) and (12)-(13). We denote the *optimal value* of problem (21) by J^{opt}.

IV. Design of Affine Control Policies

Problem (21) amounts to an infinite dimensional convex program, and is, in general, computationally intractable. We therefore resort to approximation by restricting the space of admissible control policies to those which are affine in the disturbance. In addition, we construct a polyhedral inner approximation of the feasible region of problem (21). In doing so, one can apply Proposition 3 in [25] to compute the optimal affine control policy for the resulting inner approximation through solution of a finite-dimensional conic program.

A. A Polyhedral Inner Approximation of Constraints

We derive a polyhedral inner approximation of the convex set $\mathcal{U}(\xi)$ by replacing the convex quadratic constraint (13) with the following linear constraint:

$$\left| q_i^I(t) \right| \le \overline{q_i^I}(t). \quad (22)$$

Here, the deterministic constant $\overline{q_i^I}(t)$ is defined according to

$$\overline{q_i^I}(t) = \inf \left\{ \sqrt{{s_i^I}^2 - {\xi_i^I(t)}^2} \;\middle|\; \xi \in \Xi \right\}.$$

Although such an inner approximation may seem conservative, it has been observed to result in a small loss of performance, as measured by the objective function considered in this paper. See [3], [15] for a more detailed discussion on such issues. Moreover, in Section V, we develop a technique to bound the loss of optimality incurred by this inner approximation. This loss of optimality is shown to be small for the case study considered in this paper.

Inequalities (9), (11)-(12), and (22) define a collection of linear constraints on the state, input, and disturbance trajectories. We represent them according to

$$\underline{F_x} x + \underline{F_u} u + \underline{F_\xi} \xi \le 0, \quad \forall \xi \in \Xi,$$

where the matrices $\underline{F_x} \in \mathbf{R}^{m \times N_x}, \underline{F_u} \in \mathbf{R}^{m \times N_u}, \underline{F_\xi} \in \mathbf{R}^{m \times N_\xi}$ can be specified according to the underlying problem data. The following problem is an inner approximation of problem (21):

$$\begin{array}{ll} \text{minimize} & \mathbf{E}^\gamma \left[c'x + (L_u u + L_\xi \xi)' \Sigma (L_u u + L_\xi \xi) \right] \\ \text{subject to} & \gamma \in \Gamma \\ & \left. \begin{array}{l} \underline{F_x} x + \underline{F_u} u + \underline{F_\xi} \xi \le 0 \\ x = \mathbb{A}x(0) + \mathbb{B}u \\ u = \gamma(\xi) \end{array} \right\} \forall \ \xi \in \Xi. \end{array} \quad (23)$$

Although convex, problem (23) is an infinite-dimensional robust program. Computing its optimal solution is intractable, in general. In what follows, we further approximate problem (23) from within as a finite dimensional conic program. In deriving this approximation, we restrict the space of admissible controllers to be *affine* in the disturbance.

B. Control Design via Convex Optimization

Before proceeding, we define the subspace of admissible decentralized affine control policies according to

$$S = \left\{ Q \in \mathbf{R}^{N_u \times N_\xi} \,\middle|\, Q \in \Gamma \right\},$$

where $Q \in \Gamma$ requires that the linear operator $Q : \mathbf{R}^{N_\xi} \to \mathbf{R}^{N_u}$ respects the information structure encoded in the set of admissible decentralized control policies Γ. We restrict the space of admissible controllers to be of the form $u = Q\xi$, where $Q \in S$. The optimal affine control policy for problem (23) can be computed according to Proposition 1.

Proposition 1. An optimal affine control policy for problem (23) is given by the optimal solution of the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \mathrm{Tr}\Big(\big(Q' L_u' \Sigma L_u Q + \left(2 L_\xi' \Sigma L_u + e_1 c' \mathbb{B} \right) Q \\ & \qquad + L_\xi' \Sigma L_\xi \big) M \Big) + c' \mathbb{A} x(0) \\ \text{subject to} & Q \in S \\ & Z \in \mathbf{R}^{m \times N_\xi}, \quad \Pi \in \mathbf{R}^{\ell \times m}, \quad \nu \in \mathbf{R}_+^m \\ & (\underline{F_u} + \underline{F_x} \mathbb{B}) Q + \underline{F_x} \mathbb{A} x(0) e_1' + \underline{F_\xi} + Z = 0, \\ & Z = \nu e_1' + \Pi' W, \\ & \Pi \succeq_{\mathcal{K}^*} 0, \end{array} \quad (24)$$

where $e_1 = (1, 0, \ldots, 0)$ is a unit vector in $\mathbf{R}^{N_\xi}$. Moreover, the optimal value of problem (24) equals the cost incurred by the optimal affine control policy for problem (23).

[2] Due to space constraints, the vector c, and the matrices L_u, L_ξ, and Σ are specified in the Appendix of [24].

We denote the optimal value of problem (24) by J^{in}. It clearly stands as an *upper bound* on the optimal value of problem (21); namely, $J^{\text{opt}} \leq J^{\text{in}}$.

V. A Lower Bound on Performance

Affine policies computed according to Proposition 1 are, in general, suboptimal. In this section, we bound the suboptimality incurred by such policies through the derivation of a lower bound on the optimal value of problem (21). In doing this, we construct an outer polyhedral approximation of the feasible region of problem (21), and apply Proposition 4 in [25] to derive a lower bound on the optimal value of this outer approximation.

For each $i = 1, \ldots, n$, the following linear inequality is an outer approximation of the convex quadratic constraint (13):

$$\left|q_i^I(t)\right| \leq \sqrt{2} s_i^I - \xi_i^I(t). \qquad (25)$$

Inequality (25), in combination with inequalities (9) and (11)-(12), define a collection of linear constraints on the state, input, and disturbance trajectories, which we denote by

$$\overline{F_x} x + \overline{F_u} u + \overline{F_\xi} \xi \leq 0, \quad \forall \xi \in \Xi.$$

The matrices $\overline{F_x} \in \mathbf{R}^{m \times N_x}, \overline{F_u} \in \mathbf{R}^{m \times N_u}, \overline{F_\xi} \in \mathbf{R}^{m \times N_\xi}$ can be easily specified according to the underlying problem data. With this outer approximation in hand, we can apply Proposition 4 in [25] to establish a lower bound on the optimal value of the original decentralized control design problem (21). We first require a technical assumption on the disturbance trajectory ξ.

Assumption 1. For each time $t = 0, \ldots, T-1$, there exist matrices $H_1^t, \ldots, H_n^t$, and H^t of compatible dimensions, such that

$$\mathbf{E}\left[\xi \,|\xi^t\right] = H^t \begin{bmatrix} 1 \\ \xi^t \end{bmatrix} \quad \text{and} \quad \mathbf{E}\left[\xi^t \,|\xi_i^t\right] = H_i^t \begin{bmatrix} 1 \\ \xi_i^t \end{bmatrix}$$

for all $i = 1, \ldots, n$ and $\xi \in \Xi$.

Although Assumption 1 seems restrictive, it is satisfied by a large family of distributions. We refer the readers to [26], which provides several sufficient conditions on the distribution of ξ under which Assumption 1 is satisfied.

Proposition 2 provides a lower bound on the optimal value of problem (21) through the solution of a conic program.

Proposition 2. Let Assumption 1 hold. The optimal value of the following problem is a lower bound on the optimal value of problem (21):

$$\begin{aligned} \text{minimize} \quad & \text{Tr}\Big(\big(Q' L_u' \Sigma L_u Q + \left(2 L_\xi' \Sigma L_u + e_1 c' \mathbb{B}\right) Q \\ & \quad + L_\xi' \Sigma L_\xi\big) M\Big) + c' \mathbb{A} x(0) \\ \text{subject to} \quad & Q \in S, \quad Z \in \mathbf{R}^{m \times N_\xi} \\ & (\overline{F_u} + \overline{F_x} \mathbb{B}) Q + \overline{F_x} \mathbb{A} x(0) e_1' + \overline{F_\xi} + Z = 0, \\ & W M Z' \succeq_{\mathcal{K}} 0, \\ & e_1' M Z' \geq 0, \end{aligned} \qquad (26)$$

where $e_1 = (1, 0, \ldots, 0)$ is a unit vector in $\mathbf{R}^{N_\xi}$.

We denote the optimal value of problem (26) by J^{out}. It stands as a *lower bound* on the optimal value of problem (21). In summary, we have $J^{\text{out}} \leq J^{\text{opt}} \leq J^{\text{in}}$, where J^{in} and J^{out} are calculated according to Propositions 1 and 2, respectively. A small gap between J^{in} and J^{out} implies that affine policies are close to optimal for the underlying problem instance.

VI. Case Study

We consider the control of distributed energy resources in a 12 kV radial distribution feeder, similar to the network considered in [8]. In what follows, we provide a brief description of the problem instance considered[3]. Apart from the substation, the distribution feeder consists of $n = 14$ buses, whose schematic diagram is given in Fig. 1. The voltage magnitude at bus 0 is fixed at $v_0 = 1$ per-unit (p.u.), and we require the voltage magnitude at each bus to live in the range $[0.95, 1.05]$. We operate the system over a finite time horizon of $T = 24$ hours, beginning at 12am.

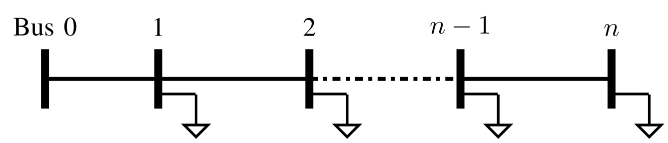

Fig. 1: Schematic diagram of a 12 kV radial distribution feeder with $n+1$ buses.

We assume that only buses 4 and 8 have storage devices and PV inverters installed. Each PV inverter ($i = 4, 8$) has an active power capacity of θ MW, and an apparent power capacity of $s_i^I = 1.25\theta$ MVA. The collection of random vectors $\{\xi_i(t)\}_{i=1,..n,\ t=0,..,T-1}$ are assumed to be mutually independent. This ensures that Assumption 1 is satisfied.

In Fig. 2, we plot the upper bound J^{in} and the lower bound J^{out}, as a function of the PV inverter active powwer capacity θ ranging from $\theta = 0$ to 2. As one might expect, the bound on the optimality gap increases with the amount of uncertain renewable supply in the distribution system. Despite this, the gap is small for all values of θ considered. This reveals that affine control policies are close to optimal.

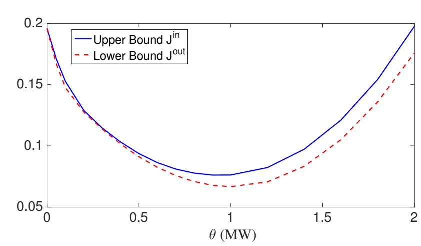

Fig. 2: The above figure plots (in MWh) the upper and lower bounds on J^{opt}, the optimal value of problem (21), as a function of the PV inverter active power capacity θ.

In Fig. 3, we plot several system trajectories at bus 8 for several independent sample paths of the disturbance process, and their empirical confidence intervals, for θ = 2. Since θ is large, the excessive active power supply from PV inverters can

[3]The exact specification of the problem data can be found in [24].

manifest in overvoltage in the distribution network. In order to ensure satisfaction of the voltage magnitude constraints, the optimal affine control policy results in reactive power injections that are negatively correlated with the active power injections from the PV inverter at bus 8. Clearly, in the absence of such a feedback control mechanism, certain realizations of the disturbance would have manifested in the violation of the voltage magnitude constraint at bus 8.

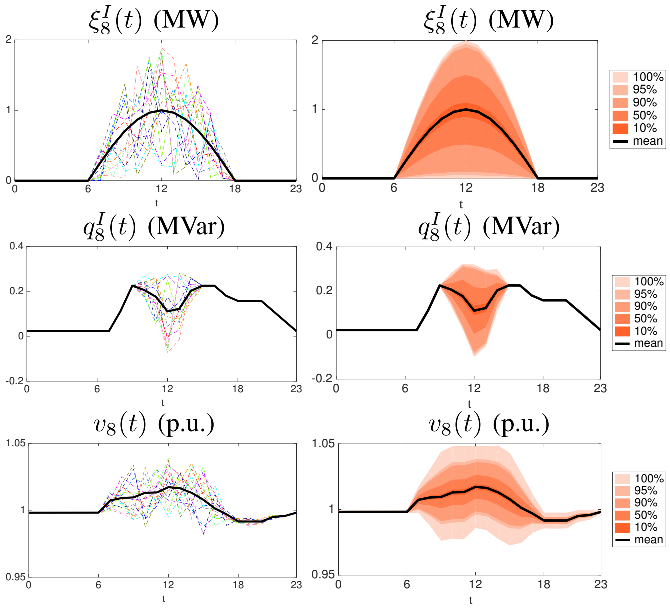

Fig. 3: The figures on the left depict the trajectories of PV inverter active and reactive power injections and voltage magnitude at bus 8. The black solid lines denote the mean trajectories. The colored dashed lines denote the realized trajectories for several independent sample paths of the disturbance process. The figures on the right depict the empirical confidence intervals for these trajectories. We set the active power capacity of PV inverters $\theta = 2$.

VII. Conclusion

There are several interesting directions for future work. For example, one potential drawback of the approach considered in this paper is the explicit reliance of the control policy on the entire disturbance history. This may incur a heavy computational burden for a long time horizon T. Thus, it will be of interest to extend the technique developed in this paper to the setting in which the control policy is restricted to have a fixed memory. It would also be of interest to explore the extent to which the selective addition of communication links between subsystems might improve the performance of the resulting decentralized controller.

References

[1] L. Cheng, Y. Chang, and R. Huang, "Mitigating voltage problem in distribution system with distributed solar generation using electric vehicles," *Sustainable Energy, IEEE Transactions on*, vol. 6, no. 4, pp. 1475–1484, 2015.

[2] *IEEE 1547 Standard for Interconnecting Distributed Resources with Electric Power Systems*, available at: http://grouper.ieee.org/groups/scc21/1547/1547_index.html.

[3] K. Turitsyn, S. Backhaus, M. Chertkov *et al.*, "Options for control of reactive power by distributed photovoltaic generators," *Proceedings of the IEEE*, vol. 99, no. 6, pp. 1063–1073, 2011.

[4] M. Farivar, L. Chen, and S. Low, "Equilibrium and dynamics of local voltage control in distribution systems," in *Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on*. IEEE, 2013, pp. 4329–4334.

[5] P. D. Ferreira, P. Carvalho, L. A. Ferreira, and M. D. Ilic, "Distributed energy resources integration challenges in low-voltage networks: Voltage control limitations and risk of cascading," *Sustainable Energy, IEEE Transactions on*, vol. 4, no. 1, pp. 82–88, 2013.

[6] N. Li, G. Qu, and M. Dahleh, "Real-time decentralized voltage control in distribution networks," in *Communication, Control, and Computing (Allerton), 2014 52nd Annual Allerton Conference on*. IEEE, 2014, pp. 582–588.

[7] B. Robbins, C. N. Hadjicostis, A. D. Domínguez-García *et al.*, "A two-stage distributed architecture for voltage control in power distribution systems," *Power Systems, IEEE Transactions on*, vol. 28, no. 2, pp. 1470–1482, 2013.

[8] H. Zhu and H. J. Liu, "Fast local voltage control under limited reactive power: Optimality and stability analysis," 2015.

[9] S. Bolognani and S. Zampieri, "A distributed control strategy for reactive power compensation in smart microgrids," *Automatic Control, IEEE Transactions on*, vol. 58, no. 11, pp. 2818–2833, 2013.

[10] E. Dall'Anese, H. Zhu, and G. B. Giannakis, "Distributed optimal power flow for smart microgrids," *IEEE Transactions on Smart Grid*, vol. 4, no. 3, pp. 1464–1475, 2013.

[11] P. Sulc, S. Backhaus, and M. Chertkov, "Optimal distributed control of reactive power via the alternating direction method of multipliers," *Energy Conversion, IEEE Transactions on*, vol. 29, no. 4, pp. 968–977, 2014.

[12] B. Zhang, A. Lam, A. D. Domínguez-García, and D. Tse, "An optimal and distributed method for voltage regulation in power distribution systems," *Power Systems, IEEE Transactions on*, vol. 30, no. 4, pp. 1714–1726, 2015.

[13] M. Farivar, R. Neal, C. Clarke, and S. Low, "Optimal inverter var control in distribution systems with high pv penetration," in *Power and Energy Society General Meeting, 2012 IEEE*. IEEE, 2012, pp. 1–7.

[14] M. Bazrafshan and N. Gatsis, "Decentralized stochastic programming for real and reactive power management in distribution systems," in *Smart Grid Communications (SmartGridComm), 2014 IEEE International Conference on*. IEEE, 2014, pp. 218–223.

[15] V. Kekatos, G. Wang, A. J. Conejo, and G. B. Giannakis, "Stochastic reactive power management in microgrids with renewables," *IEEE Transactions on Power Systems*, vol. 30, no. 6, pp. 3386–3395, 2015.

[16] K. Clement, E. Haesen, and J. Driesen, "Coordinated charging of multiple plug-in hybrid electric vehicles in residential distribution grids," in *Power Systems Conference and Exposition, 2009. PSCE'09. IEEE/PES*. IEEE, 2009, pp. 1–7.

[17] S. Deilami, A. S. Masoum, P. S. Moses, and M. A. Masoum, "Real-time coordination of plug-in electric vehicle charging in smart grids to minimize power losses and improve voltage profile," *IEEE Transactions on Smart Grid*, vol. 2, no. 3, pp. 456–467, 2011.

[18] M. Vrakopoulou, J. L. Mathieu, and G. Andersson, "Stochastic optimal power flow with uncertain reserves from demand response," in *2014 47th Hawaii International Conference on System Sciences*. IEEE, 2014, pp. 2353–2362.

[19] E. Loukarakis, C. J. Dent, and J. W. Bialek, "Decentralized multi-period economic dispatch for real-time flexible demand management," *IEEE Transactions on Power Systems*, vol. 31, no. 1, pp. 672–684, 2016.

[20] M. E. Baran and F. F. Wu, "Optimal capacitor placement on radial distribution systems," *Power Delivery, IEEE Transactions on*, vol. 4, no. 1, pp. 725–734, 1989.

[21] ——, "Optimal sizing of capacitors placed on a radial distribution system," *Power Delivery, IEEE Transactions on*, vol. 4, no. 1, pp. 735–743, 1989.

[22] ——, "Network reconfiguration in distribution systems for loss reduction and load balancing," *Power Delivery, IEEE Transactions on*, vol. 4, no. 2, pp. 1401–1407, 1989.

[23] International Electrotechnical Commission *et al.*, *Efficient Electrical Energy Transmission and Distribution*. IEC, 2007.

[24] W. Lin and E. Bitar, "Decentralized stochastic control of distributed energy resources," preprint.

[25] ——, "Performance bounds for robust decentralized control," in *2016 American Control Conference (ACC)*. IEEE, 2016, pp. 4323–4330.

[26] M. J. Hadjiyiannis, P. J. Goulart, and D. Kuhn, "An efficient method to estimate the suboptimality of affine controllers," *Automatic Control, IEEE Transactions on*, vol. 56, no. 12, pp. 2841–2853, 2011.